Advanced Optimization

Boglárka G.-Tóth and Tamás Vinkó

University of Szeged

Total Unimodular Matrices
TUM

Linear algebra

- Determinant of matrix A: $\det(A)$
- It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.
- Geometrically, it is the signed volume of the n-dimensional parallelepiped spanned by the column or row vectors of the matrix.
- The determinant is positive or negative according to whether the linear transformation preserves or reverses the orientation of a real vector space.

Linear algebra

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

$$\begin{aligned} |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei + bfg + cdh - ceg - bdi - afh. \end{aligned}$$

TUM

- Definition. A square matrix U is *unimodular* if $\det(U) = \pm 1$
- Definition. A matrix $M \in \mathbb{R}^{m \times n}$ is called *totally unimodular* if every square non-singular submatrix of M has is unimodular.

Put it differently: all submatrix U of M has $\det(U) \in \{0, 1, -1\}$.

TUM - properties

- All elements of M are either 0 or 1 or -1.
- $-M$ and M^T is also TU
- $[M \;\; I]$ is also TU
  Proof (incomplete)
  Let $\mathbf{e}_i = (0, 0, \ldots 1, 0, \ldots, 0)^T$. We are going to show that $[M \;\; \mathbf{e}_i]$ is TU.

  Choose a $k \times k$ submatrix U from M (k rows and k columns).
  – In case we have the last column and the ith row included then $\det(U) = \pm 1 \det(M^*)$, where M^* is a submatrix of M
  – In case we do not have the ith row included then $\det(U) = 0$.
  – In case we do not select the last column then we have all the columns selected from M, which is OK.

TUM - integer solution of LP

- **Theorem.** Let $M \in \mathbb{R}^{m \times n}$ (where $m < n$) be full row-rank and totally unimodular. Let $\mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{c} \in \mathbb{R}^n$.
  Then the LP:

$$\begin{aligned} & \min \mathbf{c}^T \mathbf{x} \\ \text{subject to:} \quad & M\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

  has <u>integer</u> $\mathbf{x}^* \in \mathbb{Z}^n$ solution.

This is important result as we can use any LP solver to get integer solution.
Time of solving LP: polynomial, whereas solving ILP: exponential.

TUM

- Proof. An optimal solution of an LP is a possible basis (extreme point of the polyhedron $\mathcal{P} = \{M\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$). We are going to show that these extreme points are integers.

  A vector $\mathbf{x}$ is called possible basis if

  - $M\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ which means that $\mathbf{x}$ is feasible.
  - $\mathbf{x}$ has at most m non-zero elements.
    Let $B(\mathbf{x}) \subset \{1, \ldots, n\}$ be those indices which correspond to the non-zero elements of $\mathbf{x}$.
  - The submatrix A of M which is selected by the indicies $B(\mathbf{x})$ is non-singular, i.e., $\det(A) \neq 0$. In this case the system of linear equations $A\hat{\mathbf{x}} = \mathbf{b}$ can be solved, where $\hat{\mathbf{x}}$ is a sub-vector of $\mathbf{x}$ which is selected by $B(\mathbf{x})$.

TUM

- (repeated from the previous slide):
  - The submatrix A of M which is selected by the indicies $B(\mathbf{x})$ is non-singular, i.e., $\det(A) \neq 0$. In this case the system of linear equations $A\hat{\mathbf{x}} = \mathbf{b}$ can be solved, where $\hat{\mathbf{x}}$ is a sub-vector of $\mathbf{x}$ which is selected by $B(\mathbf{x})$.

    Apply Cramer's rule:
    $$\hat{\mathbf{x}}_i = \frac{\det(A_i)}{\det(A)},$$
    where matrix A_i is obtained by changing the ith column in A into $\mathbf{b}$.
    We know that $\mathbf{b}$ is integer.
    $\det(A) = \pm 1$ for sure since matrix A is non-singular and it is a sub-matrix of M.
    $\det(A_i)$ needs to be integer.
    $\Rightarrow \hat{\mathbf{x}}_i$ is integer too $\Rightarrow \mathbf{x}^*$ is integer.

Graphs and TUMs

- Let $G = (V, E)$ be a directed graph.
- Let B the incidence matrix of G.
- B has dimension $|V| \times |E|$ and by definition

$$b_{ij} = \begin{cases} -1 & \text{if node } i \text{ is the tail of edge } j, \\ 1 & \text{if node } i \text{ is the head of edge } j, \\ 0 & \text{otherwise.} \end{cases}$$

- Example.

Graphs and TUMs

- **Theorem.** Matrix B is TU.

- Proof. By induction.

  - Assume that the theorem holds for all sub-matrices of B of size $(k-1) \times (k-1)$.

  - Take a sub-matrix U of size $k \times k$.

  - There are 3 possibilities.

    1) U has all-zero column. $\Rightarrow \det(U) = 0$.

    2) U has a column which contains a non-zero element.
    $\det(U) = \pm 1 \cdot \det(U^*)$, where U^* is a sub-matrix of size $(k-1) \times (k-1)$.

Graphs and TUMs

3) All columns of U has 2 non-zero elements.
   Within a column, one of them is $+1$ and the other one is -1.
   Hence, the sums of the columns are all equal to 0.
   In this case, the rows of the matrix are linearly dependent.
   $\Rightarrow \det(U) = 0$.

Graphs and TUMs

- Sufficient conditions: Let $A = [a_{ij}]$ be a matrix such that

  i) $a_{ij} \in \{+1, -1, 0\}$ for all i, j.

  ii) Each column contains at most two nonzero coefficients,

  $$\sum_{i=1}^{m} |a_{ij}| \leq 2 \quad (j \in [1, n]).$$

  iii) The set M of rows can be partitioned into (M_1, M_2) such that each column j containing *two* nonzero coefficients satisfies

  $$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0.$$

  Then A is totally unimodular.

Bipartite graphs and TUMs

- **Theorem.** Let G be a bipartite graph and B^+ its unsigned incidence matrix.
  Then B^+ is TU.

- Proof. Each column of B^+ contains exactly two nonzero components, a 1 for some $v \in V_1$, and a 1 for some $w \in V_2$.

  Therefore, the sufficient criterion of the above theorem applies for the choice $M_1 = V_1$, $M_2 = V_2$.

TUM - example 01

- Shortest path in directed graph G (from s to t)

- decision variable

$$x_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is part of the shortest path,} \\ 0 & \text{otherwise.} \end{cases}$$

- LP modell:

$$\min \sum_{(i,j)\in E} x_{i,j}$$

subject to

$$(B\mathbf{x})_i = \begin{cases} -1 & \text{if } i = s, \\ 1 & \text{if } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

Matrix B is the incidence matrix of G.

TUM - example 01

- Another notation:

$$\begin{aligned} & \min \mathbb{1}^T \mathbf{x} \\ \text{subject to} \quad & B\mathbf{x} = (-1, 0, 0, \ldots, 0, 1)^T \\ & \mathbf{x} \geq 0. \end{aligned}$$

- We do not need to prove that $x_{i,j} \in \{0, 1\}$ as it gets automatically fulfilled.

TUM - example 02

- Maximal pairing in bipartite graphs
- decision variable:

$$x_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is included in the pairing,} \\ 0 & \text{otherwise.} \end{cases}$$

- LP model

$$\begin{aligned} & \max \mathbb{1}^T \mathbf{x} \\ \text{subject to} \quad & B^+ \mathbf{x} \leq \mathbb{1}, \end{aligned}$$

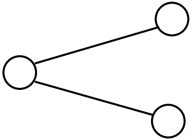
where B^+ is the unsigned incidence matrix of the bipartite graph.

- Since B^+ is TU, it is enough to have

$$\mathbf{x} \geq 0$$

as $x_{ij} \in \{0, 1\}$ holds automatically.

- The meaning of constraint $B^+\mathbf{x} \leq \mathbb{1}$:
in case we have edges as



then either the top one or the bottom one is chosen, but never together.

TUM - example 03

- Minimum $s - t$ cut