Summer School of Image Processing 2007

# THE IMAGE RESTORATION PROBLEM

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During image transmission and recording, images can be deteorate by some effects:

# Image Formation Process

During image transmission and recording, images can be deteorate by some effects:

- deterministic effects: blur

During image transmission and recording, images can be deteorate by some effects:

- deterministic effects: blur
- random effects: noise

# NOISE EFFECTS

$$\begin{pmatrix} 200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \\ 200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \\ 200 & 200 & 50 & 50 & 50 & 50 & 200 & 200 \\ 200 & 200 & 50 & 50 & 50 & 50 & 200 & 200 \\ 200 & 200 & 50 & 50 & 50 & 50 & 200 & 200 \\ 200 & 200 & 50 & 50 & 50 & 50 & 200 & 200 \\ 200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \\ 200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \end{pmatrix}$$

$$
\begin{pmatrix}
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210 & 175 & 200 & 199 & 200 & 192 & 227 & 153
\end{pmatrix}
$$

# Noise Effects

$$
\begin{pmatrix}
200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \\
200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \\
200 & 200 & 50 & 50 & 50 & 50 & 200 & 200 \\
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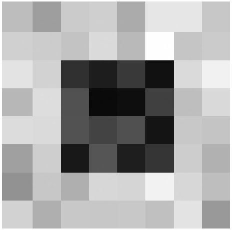
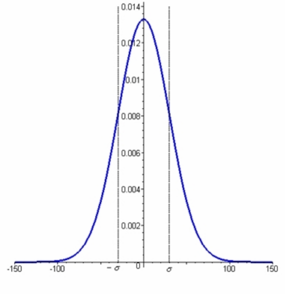
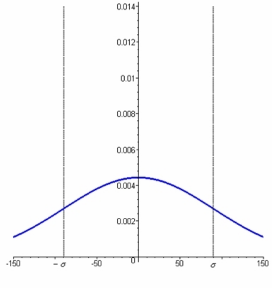


# Probability Distribution of the Noise

$$p(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}$$



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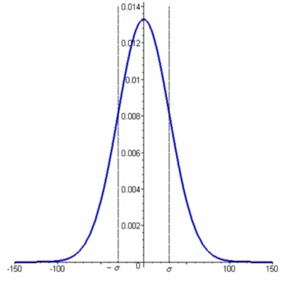


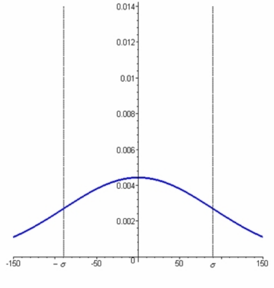


small $\sigma^2 \Longrightarrow$ light noise

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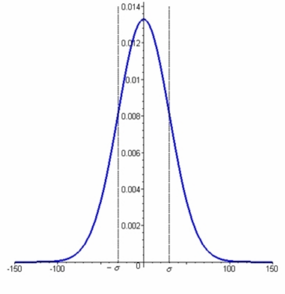


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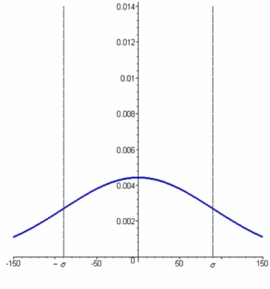


large $\sigma^2 \Longrightarrow$ heavy noise

# Noisy Images

*Ideal image*

# Noisy Images

$\sigma^2 = 100$

# NOISY IMAGES

$\sigma^2 = 400$

# NOISY IMAGES

$\sigma^2 = 900$

# Noisy Images

$\sigma^2 = 1600$

$$M = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 10 & 5 \\ 1 & 5 & 1 \end{pmatrix}$$

*Noisy image*

# Blur Effects

$$M = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 10 & 5 \\ 1 & 5 & 1 \end{pmatrix}$$

*Blurred and noisy image*

# Blur Effects

$$\begin{pmatrix} 67 & 69 & 75 & 207 & 201 \\ 69 & 74 & 69 & 71 & 199 \\ 69 & 68 & 74 & 69 & 71 \\ 68 & 70 & 69 & 65 & 78 \\ 69 & 71 & 74 & 65 & 71 \\ 67 & 70 & 68 & 69 & 72 \end{pmatrix} \quad \begin{pmatrix} *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \end{pmatrix}$$

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$$M = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 10 & 5 \\ 1 & 5 & 1 \end{pmatrix}$$

$$x(3,3) = \frac{74\times1+69\times5+71\times1+68\times5+74\times10+69\times5+70\times1+69\times5+65\times1}{1+5+1+5+10+5+1+5+1}$$

# Blur Effects

$$\begin{pmatrix} 67 & 69 & 75 & 207 & 201 \\ 69 & 74 & 69 & 71 & 199 \\ 69 & 68 & 74 & 69 & 71 \\ 68 & 70 & 69 & 65 & 78 \\ 69 & 71 & 74 & 65 & 71 \\ 67 & 70 & 68 & 69 & 72 \end{pmatrix} \quad \begin{pmatrix} *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & 70 & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \\ *** & *** & *** & *** & *** \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 10 & 5 \\ 1 & 5 & 1 \end{pmatrix}$$

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$$M = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 10 & 5 \\ 1 & 5 & 1 \end{pmatrix}$$

$$x(1,1) = \frac{67\times10+69\times5+69\times5+74\times1}{10+5+5+1}$$

$$
\begin{pmatrix}
67 & 69 & 75 & 207 & 201 \\
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68 & *** & *** & *** & *** \\
*** & *** & *** & *** & *** \\
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$$

$$
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$$

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$$M = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 10 & 5 \\ 1 & 5 & 1 \end{pmatrix}$$

$$x(1,2) = \frac{67\times 5+69\times 10+75\times 5+69\times 1+74\times 5+69\times 1}{5+10+5+1+5+1}$$

# BLUR EFFECTS

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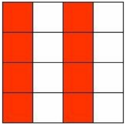
$$x(1,2) = \frac{67\times 5+69\times 10+75\times 5+69\times 1+74\times 5+69\times 1}{5+10+5+1+5+1} = 70$$

# Lexicographical Notation

Using the lexicographical notation it is possible to consider an image as a vector



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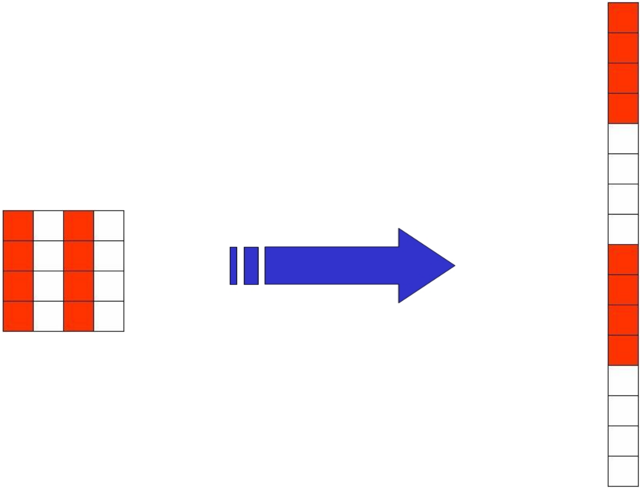

# Mathematical Model

$$Ax + n = y$$

$x \in \mathbb{R}^{n \times m}$ original image
$y \in \mathbb{R}^{n \cdot m}$ observed image
$n \in \mathbb{R}^{n \cdot m}$ Gaussian noise with zero mean and variance $\sigma^2$
$A \in \mathbb{R}^{(n \cdot m) \times (n \cdot m)}$ linear blur operator


*Original image* $x$


*Observed image* $Ax$

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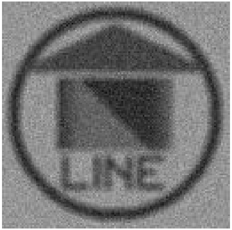


*Observed Image* $Ax + n$

# The Image Restoration Problem

## Inverse Problem

The image restoration problem consists of finding an estimation of the original image $\boldsymbol{x}$, given the blur matrix $A$, the observed image $\boldsymbol{y}$ and the variance $\sigma^2$ of the noise.

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*Obversed image*

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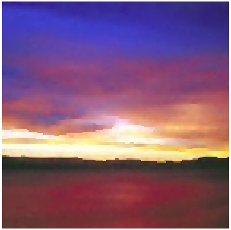

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### BAD NEWS

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That is one of the following conditions on the solution does not hold:

- existence
- uniqueness
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# Ill–Posedness of the Problem

### Bad news

The image restoration problem is ill–posed in the sense of Hadamard.

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- stabiliy

Solution: regularization of the problem

By a regularization technique we can impose the following constraints on the solution:

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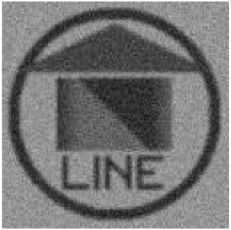
- data consistancy
- regularity
- presenving of discontinuities
- adjacent parallel lines inhibition

# DATA CONSISTANCY CONSTRAINT



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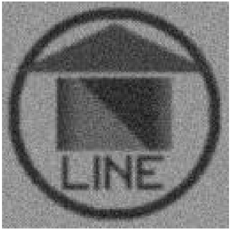


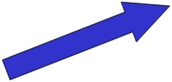

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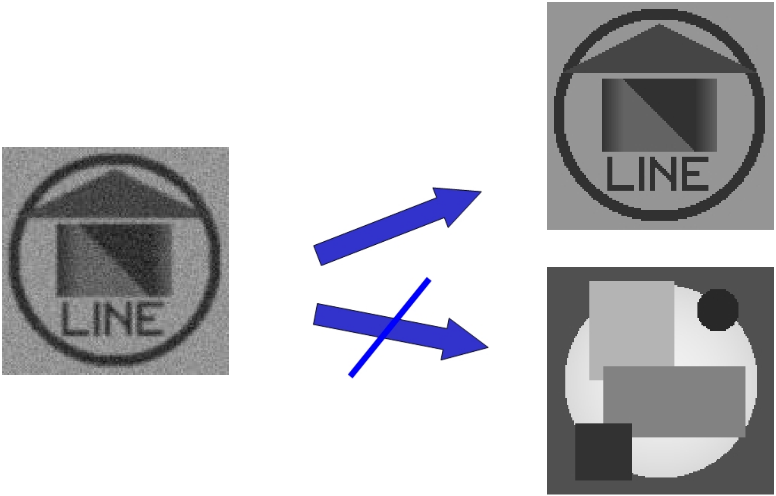

# Regularity Constraint

$$
\begin{pmatrix}
189 & 158 & 203 & 207 & 171 & 230 & 230 & 199 \\
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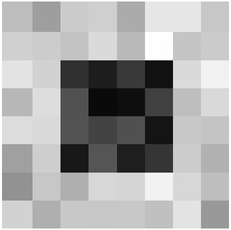

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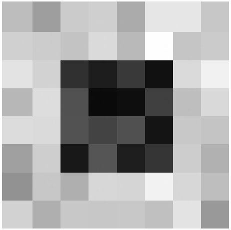
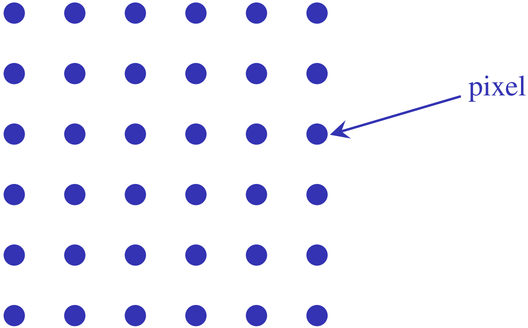


# Edge–Presenving Constraint



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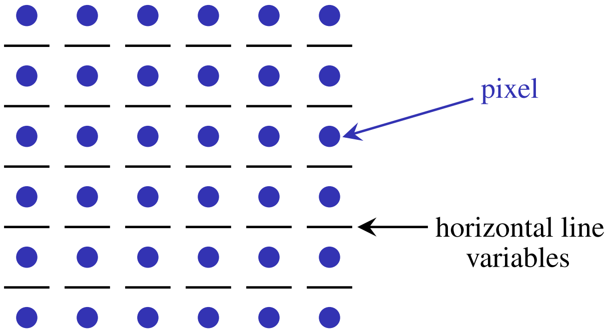

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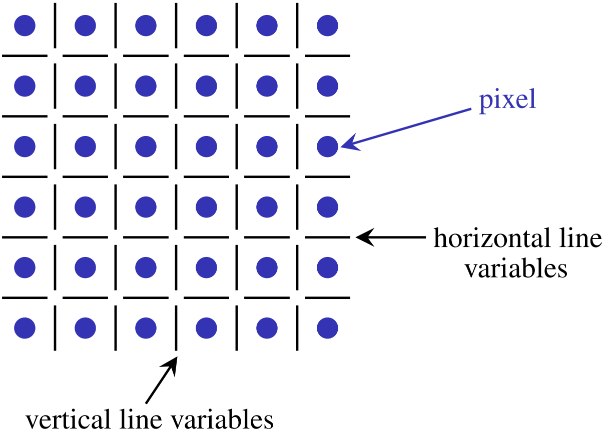

# Adjacent Parallel Lines Inhibition


*Restored image*


*Restored Image*

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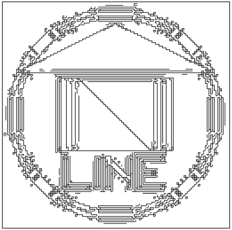


*Image edges*

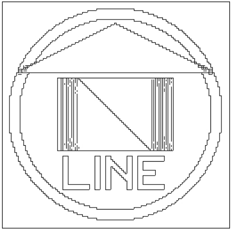


*Image edges*

# First Order Cliques

(1) $\begin{array}{ll} s & \bullet \\ t & \bullet \end{array}$ (2) $s \;\bullet \qquad t \;\bullet$

Associated finite order operator:

$$D_c^1 \boldsymbol{x} = x_s - x_t, \qquad \forall\; c \text{ of kind } (1) \text{ and } (2),$$

$$C_1 = \{c | c \text{ is a first order clique}\}.$$

# Second Order Cliques

$$(1) \quad \begin{array}{cc} s & \bullet \\ t & \bullet \\ r & \bullet \end{array} \qquad\qquad (2) \quad s \;\bullet\;\; t \;\bullet\;\; r \;\bullet$$

Associated finite order operator:

$$D_c^2 \boldsymbol{x} = \; x_s - 2x_t + x_r, \qquad \forall\; c \text{ of kind } (1) \text{ and } (2),$$

$$C_2 = \{c | c \text{ is a second order clique}\}.$$


*Observed image*


$1^o$ *order: MSE=11.439*


$2^o$ *order: MSE=10.2368*

# Edge–Preserving Regularization Technique

## Definition

The solution of the problem can be defined as the argument of the minimum of the following primal energy function:

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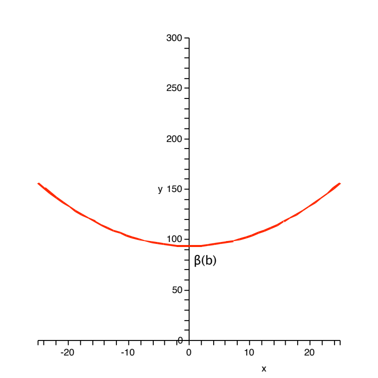

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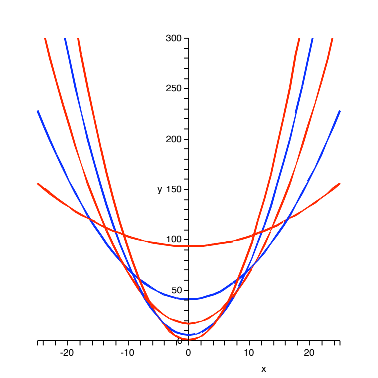

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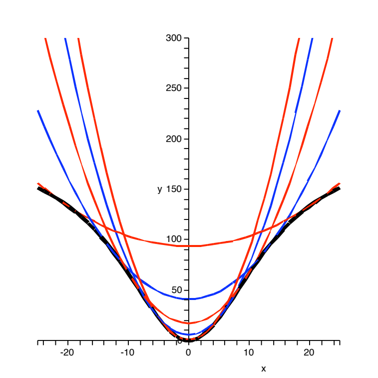
$$b \in B = (0, 1] \quad \beta(b) = \alpha(1 - 2\sqrt{b} + b) \implies \quad g(t) = \frac{\lambda^2 t^2}{\frac{\lambda^2}{\alpha} t^2 + 1}$$



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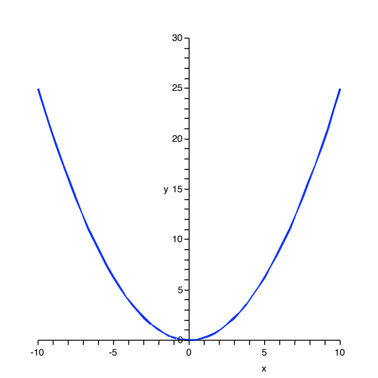

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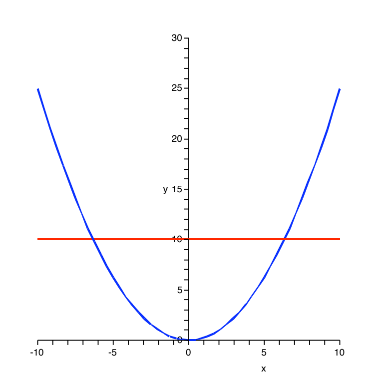

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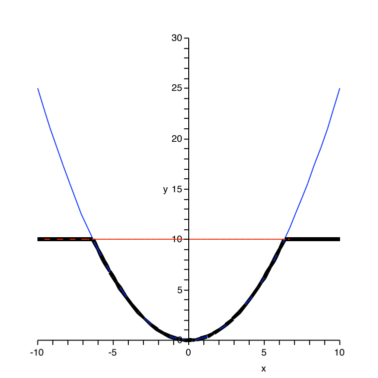
$$b \in B = \{0, 1\} \quad \beta(b) = \alpha(1 - b) \implies \quad g(t) = \min\{\lambda^2 t^2, \alpha\}$$



$$\boxed{B,\, \beta} \qquad \Longleftrightarrow \qquad \boxed{g}$$

$B$ is the set of line variable values
$\beta$ is the function in the primal energy
$g$ is the function in the dual energy

# Minimization of the Dual Energy

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### BAD NEWS

The dual energy function is not convex

The classical Simutaled Annealing algorithm:

put $\boldsymbol{x}$ as initial image;

put $T = T_0$ as initial temperature;

while $T \neq 0$ do

generate a new image
cumpute the difference of energy $\Delta E_d$;
if $\Delta E_d \leq 0$, accept the new image as $\boldsymbol{x}$;
if $\Delta E_d > 0$ accept the new image as $\boldsymbol{x}$ with
probability equal to $e^{\frac{\Delta E_d}{T}}$;
decrease the temperature $T$

The GNC (*Graduated Non–Convexity*) technique requires to find a finite family of approximating functions $\{E_d^{(p_\kappa)}\}_{\kappa\in\{1,\dots,\bar{\kappa}\}}$, such that the first $E_d^{(p_1)}$ is convex and the last $E_d^{(p_{\bar{\kappa}})} = E_d$ is the original dual energy function.

# Deterministic Algorithm

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Hence the following algorithm is applied from an initial point $\boldsymbol{x}_0$:

$\kappa = 1$;<br>
while $\kappa \neq \bar{\kappa}$ do<br>
    $\boldsymbol{x}_\kappa$ is equal to the stationary point among the speediest descent direction of $E_d^{(p_\kappa)}$, starting from $\boldsymbol{x}_{\kappa-1}$;<br>
    $\kappa = \kappa + 1$;

The blind image separation problem consist of findind the source images from some mixtures of them

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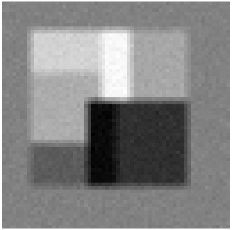
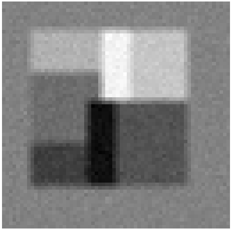


*First mixture*



*Second mixture*

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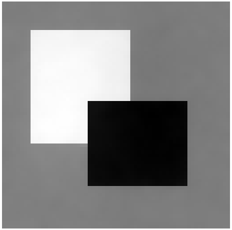

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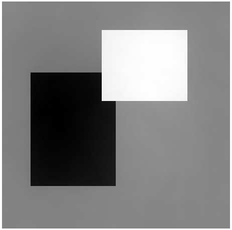

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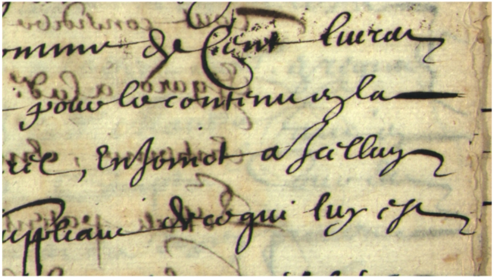
# Document Restoration



*First mixture*

*Second mixture*

*Restored first source*

*Restored second source*

# The Blind Image Separation Problem

*First mixture*


*Second mixture*

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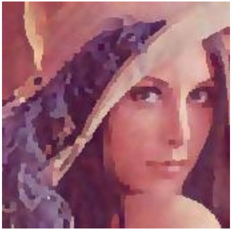
*Restored first source*


*Restored second source*

*First mixture*

*Second mixture*

# THE BLIND IMAGE SEPARATION PROBLEM



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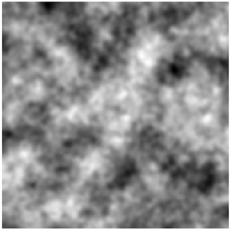 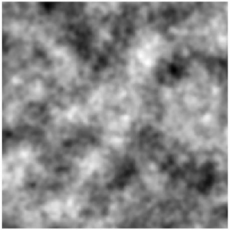 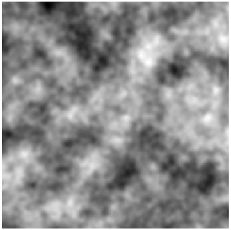

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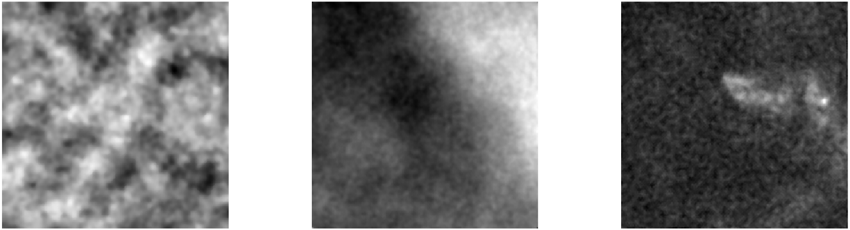

# THE END