FOURIER TRANSFORMATION AND ITS APPLICATIONS IN IMAGE PROCESSING

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Contents

Historical introduction

Jean-Baptiste Joseph Fourier 1768-1830

taught mathematics in Paris
eventually traveled to Egypt with Napoleon to become the secretary of the Institute of Egypt
after fall of Napoleon worked at Bureau of Statistics
elected to National Academy of Sciences in 1817

La Theorie Analytique de la Chaleur (The Analytic Theory of Heat), 1822
revolutionary ideas about how to solve a class of linear differential equations

Fourier: The Analytic Theory of Heat

– any periodic function can be represented by the sum of sinusoids and cosines: now called the Fourier Series;

– any curve, that eventually repeats itself, no matter how complex, can be expressed as the sum of smoothly oscillating functions;

– approach: multiply the sinusoids and cosines by coefficients to change their amplitude, shift them so that that either add or cancel one another (changing phase);

– non-periodic functions, those that have finite area, can also be represented in this way

Mathematics

Fourier transformation (FT)

$f \xrightarrow{\mathcal{F}} F$

$$[\mathcal{F}f](X) = F(X) = \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i x X} dx$$

inverse Fourier transformation (IFT)

$f \xleftarrow{\mathcal{F}^{-1}} F$

$$[\mathcal{F}^{-1}F](x) = f(x) = \int_{-\infty}^{\infty} F(X) \cdot e^{2\pi i x X} dX$$

$f \xrightarrow{\mathcal{F}} F$ uniquely determined*

Mathematics

the FT of a function is usually complex:

$$F(X) = \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i x X} dx = \int_{-\infty}^{\infty} f(x) \cdot [\cos(2\pi x X) - i \cdot \sin(2\pi i x X)] dx =$$

$$= \underbrace{\int_{-\infty}^{\infty} f(x) \cdot \cos(2\pi x X) dx}_{real \quad part} - i \cdot \underbrace{\int_{-\infty}^{\infty} f(x) \cdot \sin(2\pi x X) dx}_{imaginary \quad part}$$

$i = \sqrt{-1}$

Mathematics

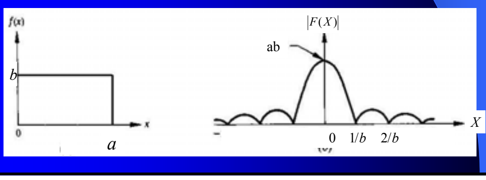
an example:

$$f(x) = \begin{cases} b, & \text{if } 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$F(X) = \int_0^a b \cdot e^{-2\pi i x X} dx = b \cdot \frac{-1}{2\pi i X} \cdot e^{-2\pi i x X} \Big|_0^a =$$

$$= \frac{b}{2\pi i X}(1 - e^{-2\pi i a X}) = \frac{b \cdot e^{-\pi i b X}}{2\pi i X}(e^{\pi i b X} - e^{-\pi i b X}) =$$

$$= \frac{b \cdot e^{-\pi i b X}}{\pi X} \cdot \sin(\pi b X)$$

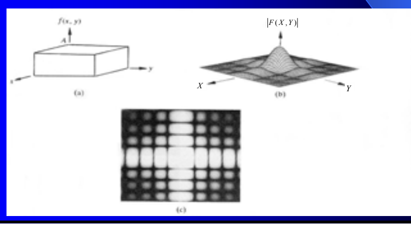


2-dimensional case

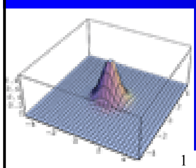
$$F(X,Y) = \int\int_{-\infty}^{\infty} f(x,y) \cdot e^{-2\pi i (xX + yY)} dxdy$$

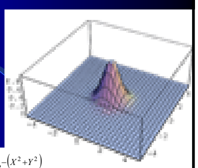
$$f(x,y) = \int\int_{-\infty}^{\infty} F(X,Y) \cdot e^{2\pi i (xX + yY)} dxdy$$

an example

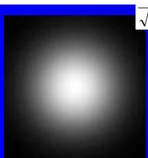


Properties





$$\frac{1}{\sqrt{2\pi}} e^{-(x^2+y^2)} \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{2\pi}} e^{-(X^2+Y^2)}$$



the FT of a Gaussian is
another Gaussian,
i.e., *eigenfunction*

Sums of sinusoids

inverse Fourier transformation (IFT)

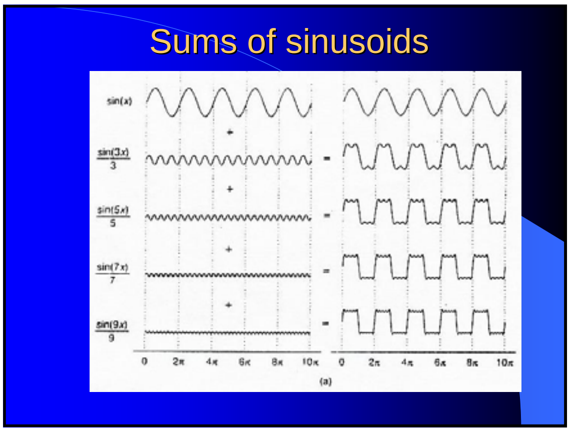
$f \xleftarrow{\mathcal{F}^{-1}} F$

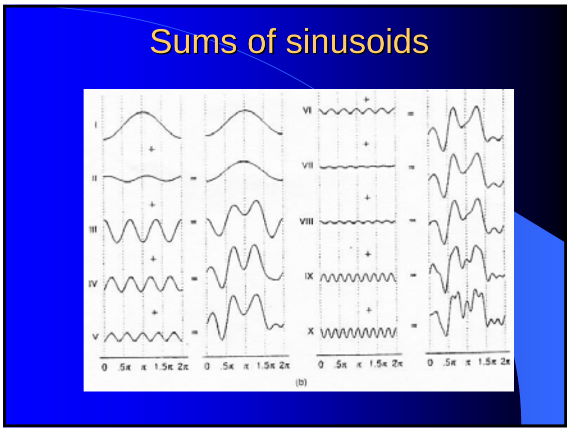
$$f(x) = \int_{-\infty}^{\infty} F(X) \cdot e^{2\pi i x X} dX$$

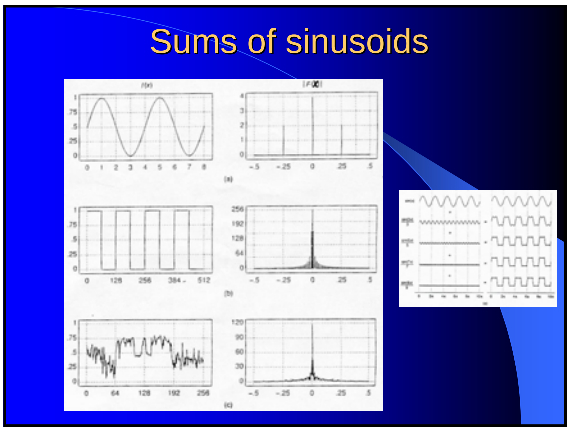
an interpretation:

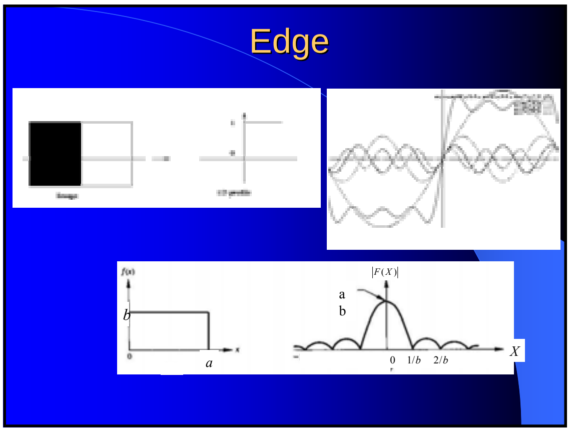
Any periodic function can be decomposed into a series of sinusoidal waveforms of various frequencies and amplitudes.

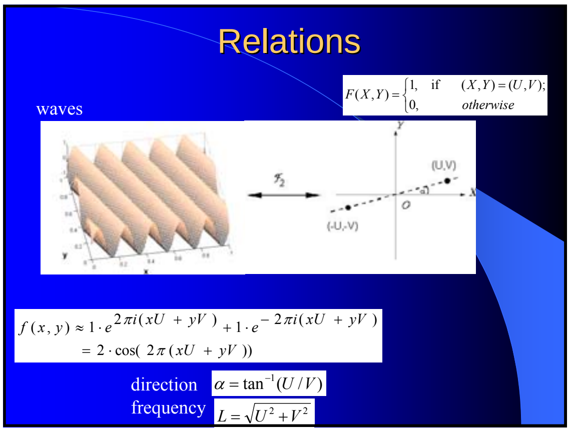
$$f(x) = \int_{-\infty}^{\infty} F(X) \cdot \cos(2\pi x X) dX + i \cdot \int_{-\infty}^{\infty} F(X) \cdot \sin(2\pi x X) dX$$

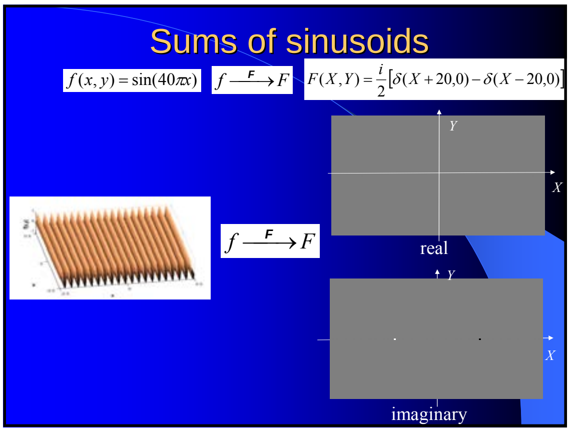

Sums of sinusoids


Sums of sinusoids


Sums of sinusoids


Edge

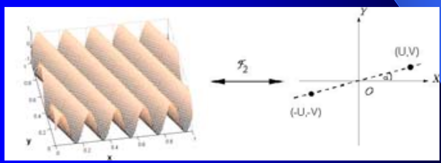

Relations
waves
$F(X,Y)=\begin{cases}1, & \text{if } (X,Y)=(U,V);\\ 0, & otherwise\end{cases}$
(U,V)
(-U,-V)
$f(x,y) \approx 1 \cdot e^{2\pi i(xU+yV)} + 1 \cdot e^{-2\pi i(xU+yV)}$
$= 2 \cdot \cos(2\pi(xU+yV))$
direction $\alpha = \tan^{-1}(U/V)$
frequency $L=\sqrt{U^2+V^2}$


Sums of sinusoids
$f(x,y)=\sin(40\pi x)$
$f \xrightarrow{F} F$
$F(X,Y)=\frac{i}{2}[\delta(X+20,0)-\delta(X-20,0)]$
real
imaginary

Waves, points, and frequencies

points in the image $F(X,Y)$ represent the contribution of frequency (X,Y) to the original image $f(x,y)$

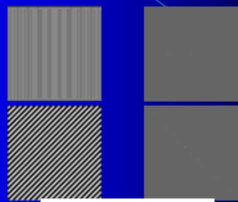
The Fourier transformation determines the magnitude (amplitude – $|F(X,Y)|$) of each possible frequency (X,Y).



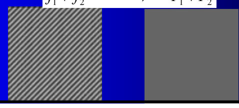
Linearity

$$a_1 \cdot f_1 + a_2 \cdot f_2 \xrightarrow{F} a_1 \cdot F_1 + a_2 \cdot F_2$$

example



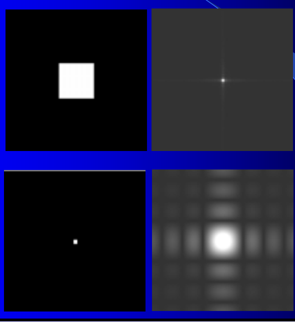
$$f_1 + f_2 \xrightarrow{F} F_1 + F_2$$



Scaling

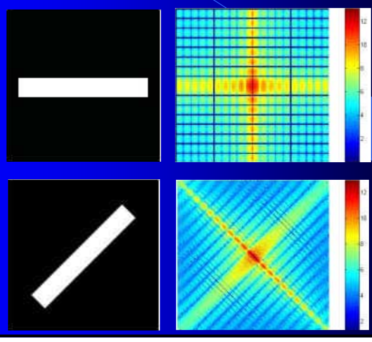
$$f(ax,by) \xrightarrow{F} \frac{1}{|ab|} F\left(\frac{X}{a},\frac{Y}{b}\right)$$

example



Rotation

example



Discrete Fourier Transformation DFT

Discrete Fourier Transformation

DFT and IDFT:

$$f_0, f_1, \ldots, f_{N-1} \xrightarrow{F} F_0, F_1, \ldots, F_{N-1}$$

$$F_l = \sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi}{N}kl}, \quad l = 0,1,\ldots,N-1$$

$$f_k = \frac{1}{N}\sum_{l=0}^{N-1} F_l e^{i\frac{2\pi}{N}kl}, \quad k = 0,1,\ldots,N-1$$

2D DFT and IDFT:

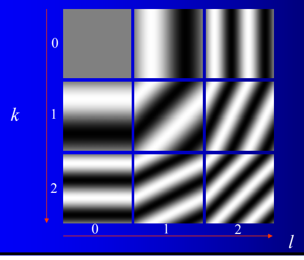
$$F_{k,l} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f_{m,n} \cdot e^{-i\frac{2\pi}{N}(mk+nl)}, \quad k,l = 0,1,\ldots,N-1$$

$$f_{m,n} = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l} \cdot e^{i\frac{2\pi}{N}(mk+nl)}, \quad m,n = 0,1,\ldots,N-1$$

2D Discrete Fourier Transformation

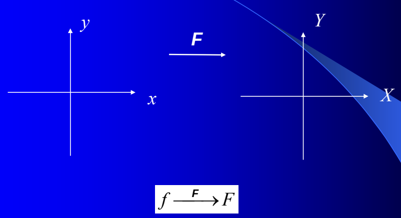
$$f_{m,n} = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_{k,l}\cdot e^{i\frac{2\pi}{N}(mk+nl)}$$

f is the sum of so-called *base functions*:



Fourier transformation of images

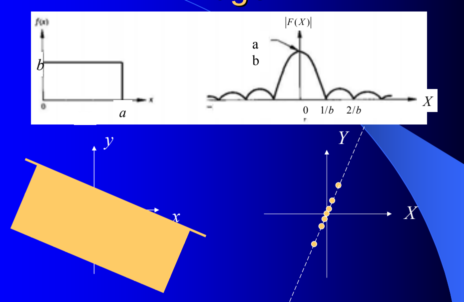
Image and frequency spaces



normal space
image space

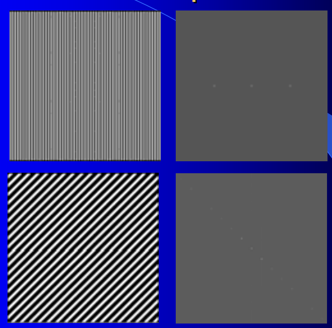
Fourier space
frequency space

Edge



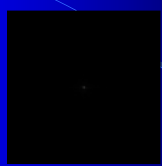
there is a line being perpendicular to the direction of the edge

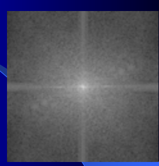
Example



Example

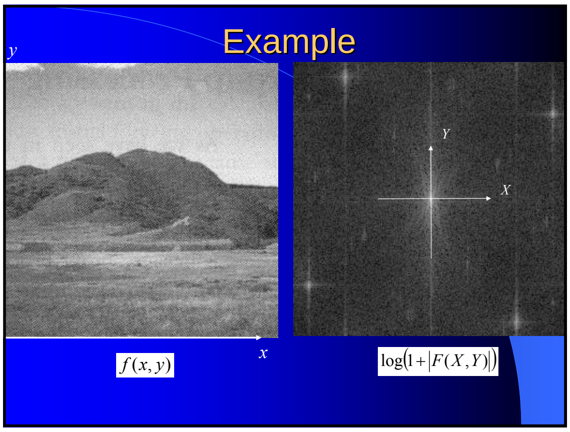

$f(x,y)$

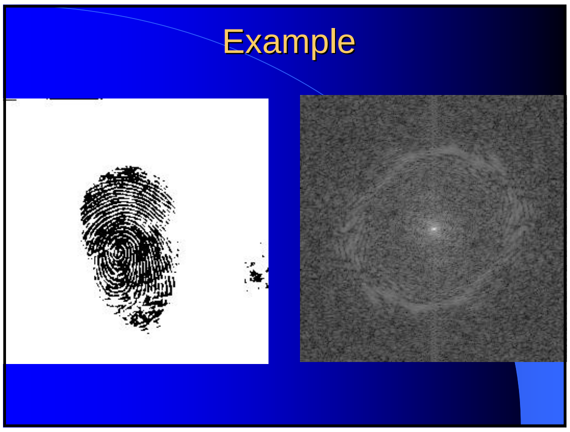

$|F(X,Y)|$

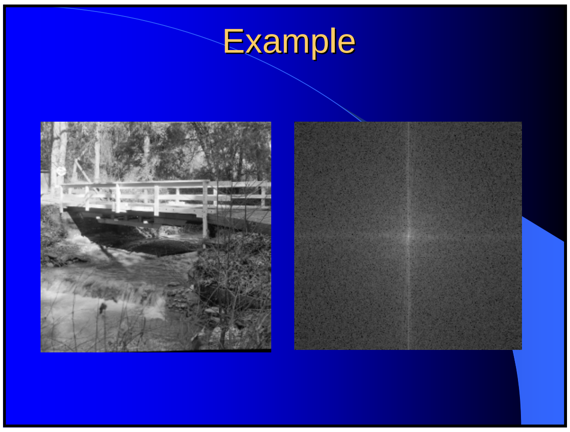

$\log(1+|F(X,Y)|)$

$F(0,0)$-value is by far the largest component of the image, other frequency components are usually much smaller, the magnitude of $F(X,Y)$ decreases quickly

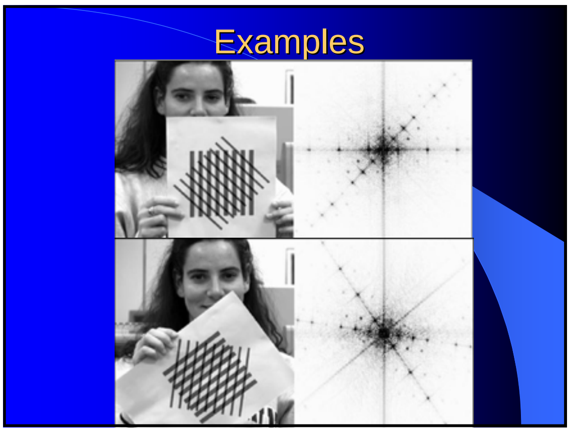
Fisher, Perkins, Walker, Wolfart, 1994

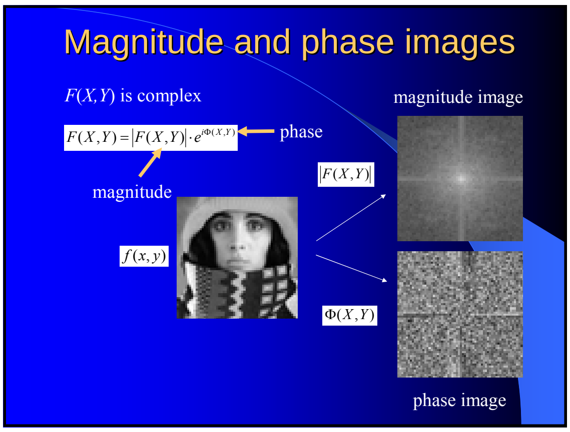

Example
y
Y
X
x
f(x,y)
log(1+|F(X,Y)|)


Example

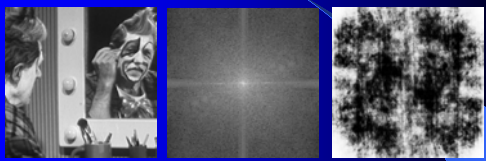

Example


Example


Examples


Magnitude and phase images
F(X,Y) is complex
magnitude image
$F(X,Y)=|F(X,Y)|\cdot e^{i\Phi(X,Y)}$
phase
$|F(X,Y)|$
magnitude
$f(x,y)$
$\Phi(X,Y)$
phase image

Phase image



$f(x,y)$ $\log(1+|F(X,Y)|)$ $F^{-1}(|F(X,Y)|)$

the phase information is also important

Fisher, Perkins, Walker, Wolfart, 1994

Filtering

Convolution

$$[f*g](x)=\int_{-\infty}^{\infty} f(\xi)\cdot g(x-\xi)d\xi \quad \text{1D}$$

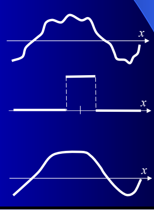
$$[f*g](x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot g(x-\xi,y-\eta)d\xi\eta \quad \text{2D}$$

example: smoothing

$f(x)$

$$g(x)=\begin{cases}1, & \text{if } |x|<1/2,\\ 0, & \text{otherwise}\end{cases}$$

$$[f*g](x)=\int_{x-1/2}^{x+1/2} f(\xi)d\xi$$



Convolution theorem

$$f*h \xrightarrow{\mathbf{F}} F\cdot H$$

$$f\cdot h \xrightarrow{\mathbf{F}} F*H$$

Filtering

(in the Fourier space) multiplication of F with a filter function H : $F{\cdot}H$

Convolution theorem:

$$f*h \xrightarrow{\mathbf{F}} F\cdot H$$

$$h=\frac{1}{l^2}\begin{pmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{pmatrix}_{l\times l} \xrightarrow{\mathbf{F}}$$

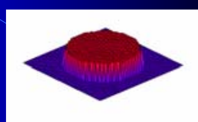
$$H(j,k)=\frac{1}{N}\sum_{m=0}^{l-1}\sum_{n=0}^{l-1}1\cdot e^{-\frac{2\pi i}{N}(mj+nk)}=$$

$$=\frac{1}{N}\left(\sum_{m=0}^{l-1}e^{-\frac{2\pi i}{N}mj}\right)\cdot\left(\sum_{n=0}^{l-1}e^{-\frac{2\pi i}{N}nk}\right)=$$

$$=\frac{1}{N}e^{i\pi(l-1)(j+k)}\frac{\sin(\pi lj)}{\sin(\pi j)}\cdot\frac{\sin(\pi kl)}{\sin(\pi k)}$$

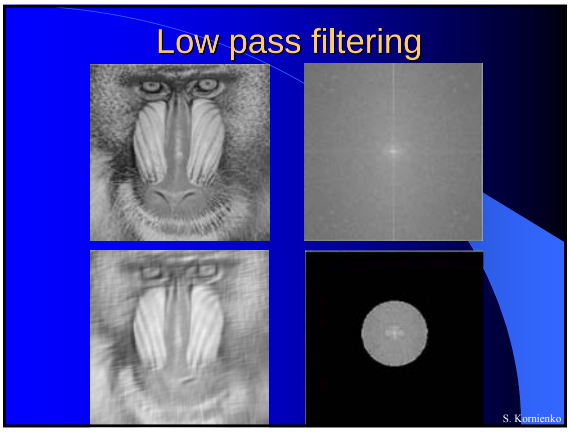
Ideal low-pass filter (ILPF)

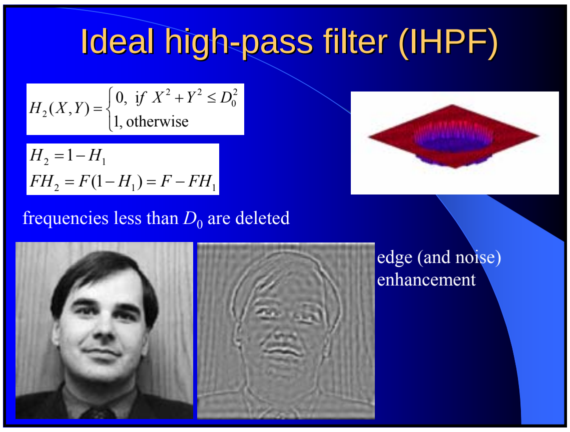
$$H_1(X,Y)=\begin{cases}1, \text{ if } X^2+Y^2\le D_0^2\\ 0, \text{ otherwise}\end{cases}$$

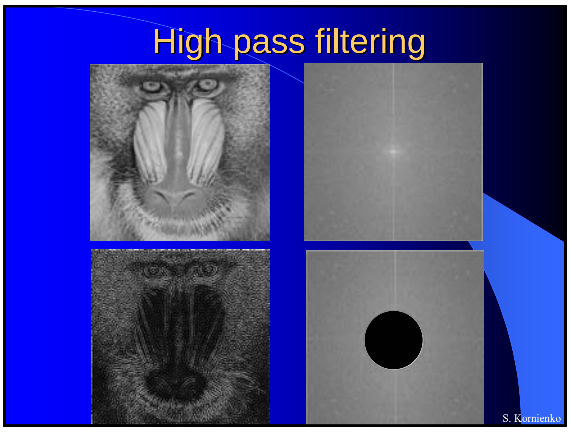


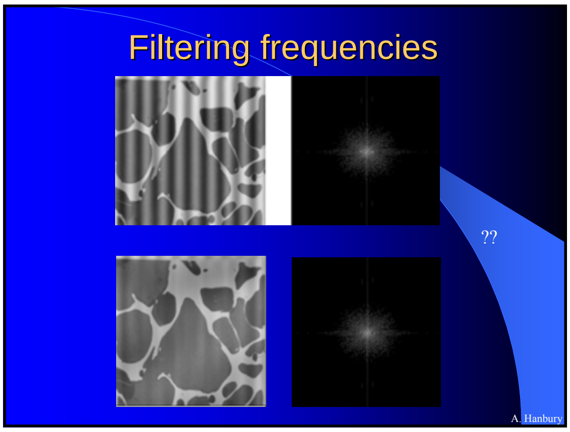
frequencies greater than D_0 are deleted

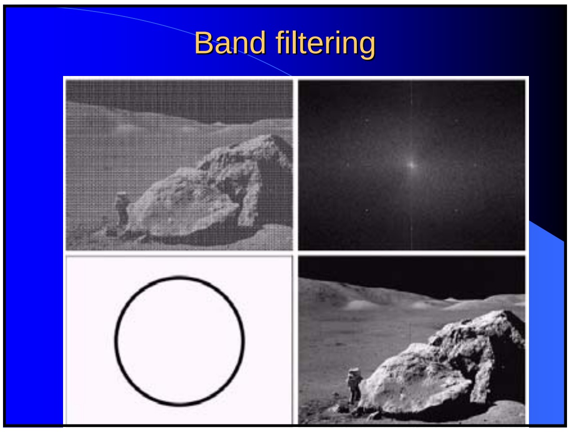
smoothing

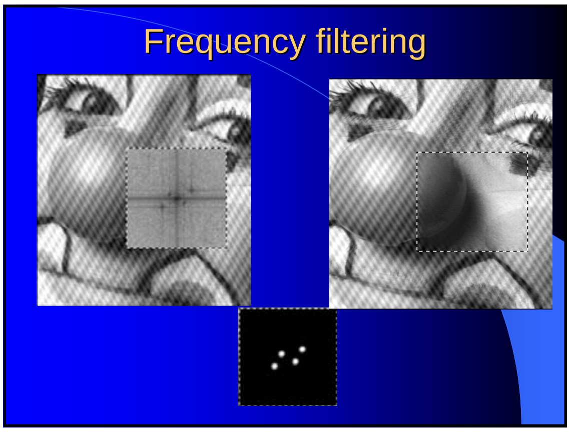

Low pass filtering
S. Kornienko

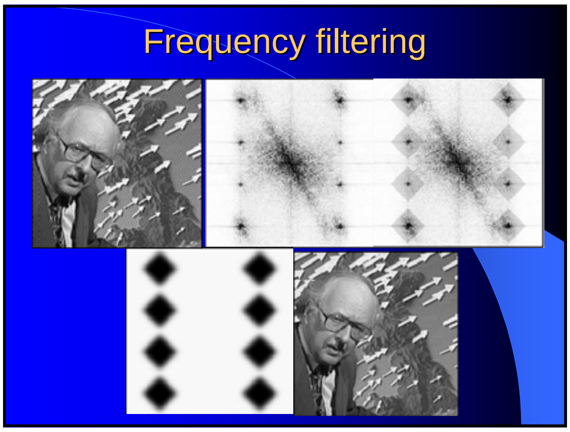

Ideal high-pass filter (IHPF)
$H_2(X,Y)=\begin{cases}0, & \text{if } X^2+Y^2\le D_0^2\\ 1, & \text{otherwise}\end{cases}$
$H_2 = 1 - H_1$
$FH_2 = F(1-H_1) = F - FH_1$
frequencies less than D_0 are deleted
edge (and noise) enhancement

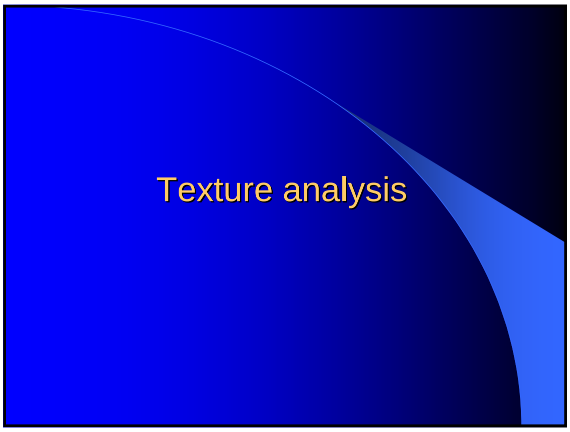

High pass filtering
S. Kornienko

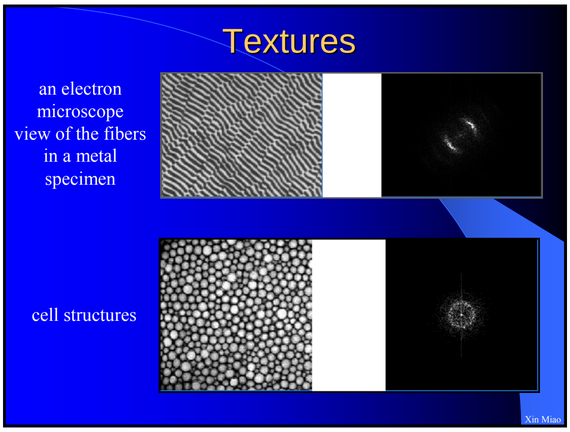

Filtering frequencies
??
A. Hanbury

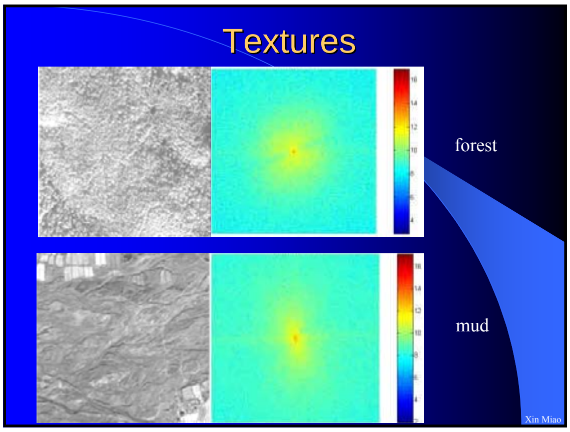

Band filtering

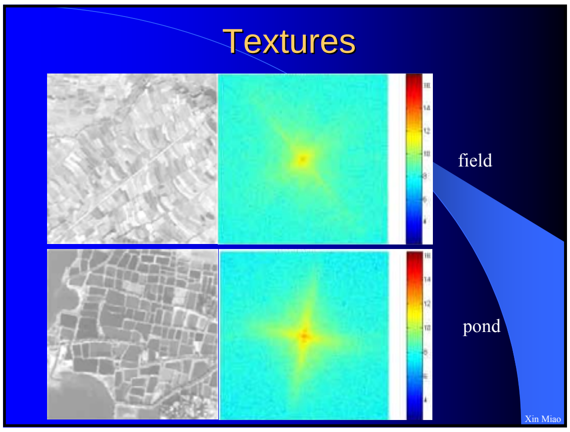

Frequency filtering

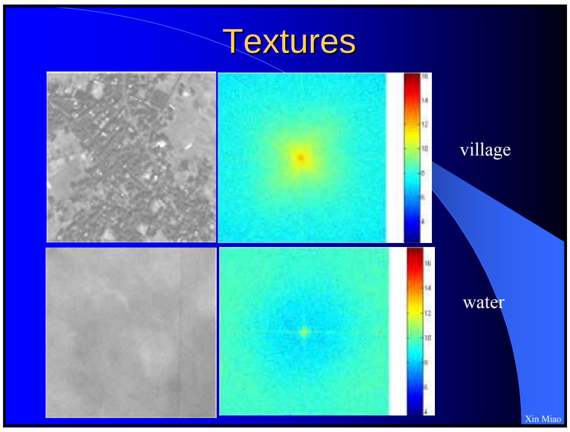

Frequency filtering

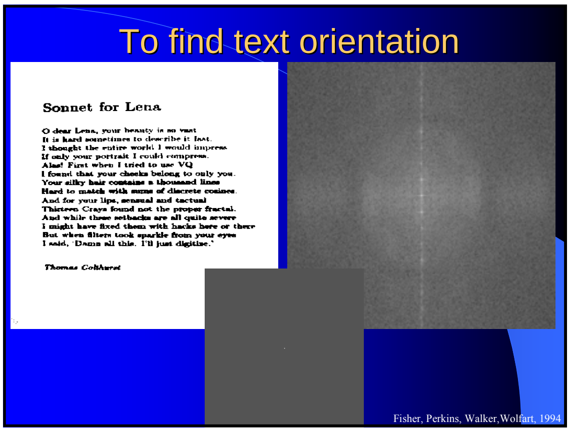

Texture analysis

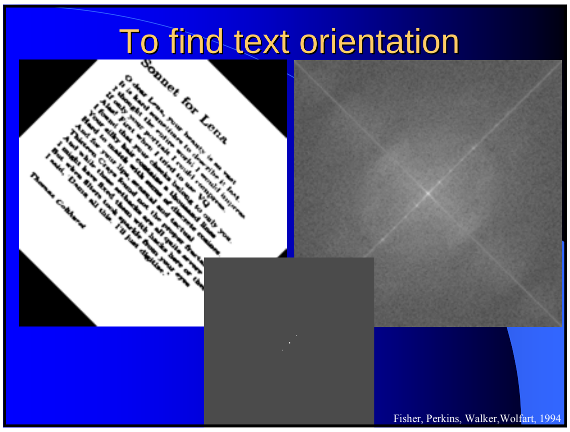

Textures
an electron microscope view of the fibers in a metal specimen
cell structures
Xin Miao

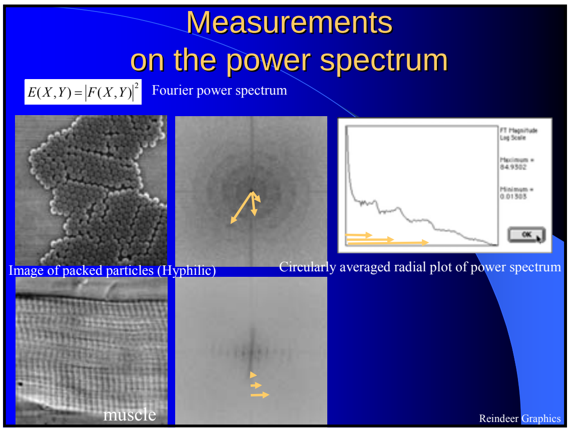

Textures
forest
mud
Xin Miao

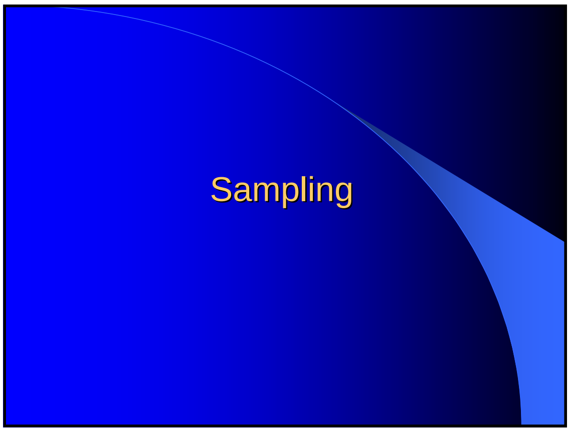

Textures
field
pond
Xin Miao

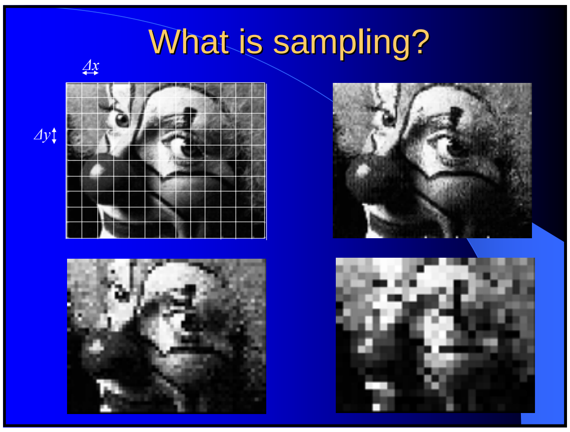

Textures
village
water
Xin Miao

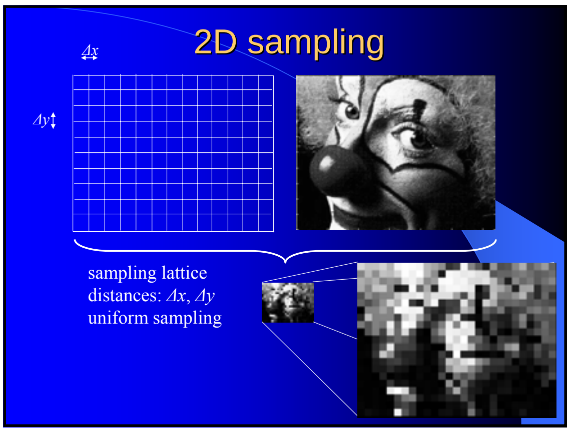

To find text orientation
Sonnet for Lena
Thomas Colthurst
Fisher, Perkins, Walker,Wolfart, 1994

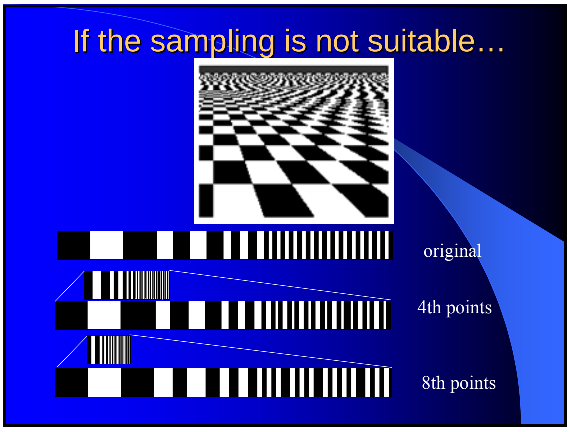

To find text orientation
Sonnet for Lena
Fisher, Perkins, Walker,Wolfart, 1994

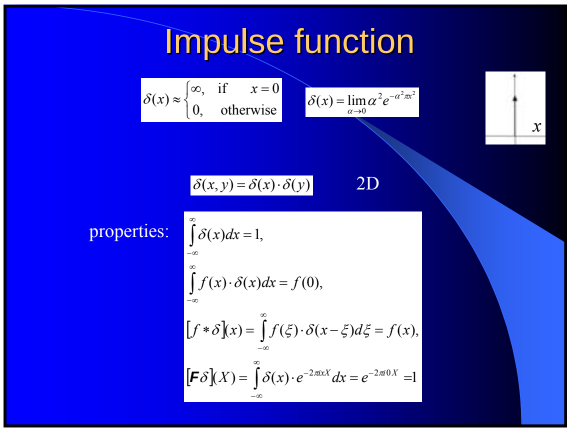

Measurements
on the power spectrum
$E(X,Y)=|F(X,Y)|^2$
Fourier power spectrum
Image of packed particles (Hyphilic)
Circularly averaged radial plot of power spectrum
muscle
Reindeer Graphics

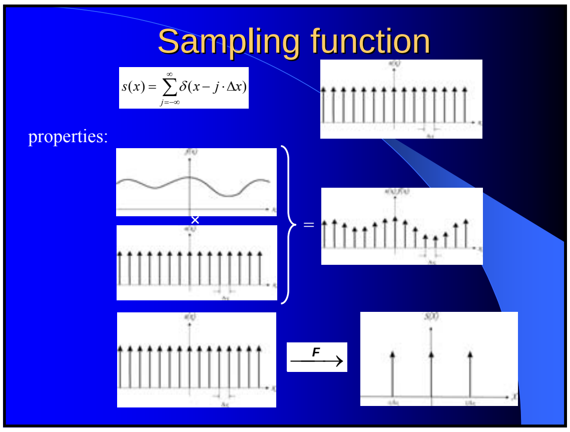

Sampling

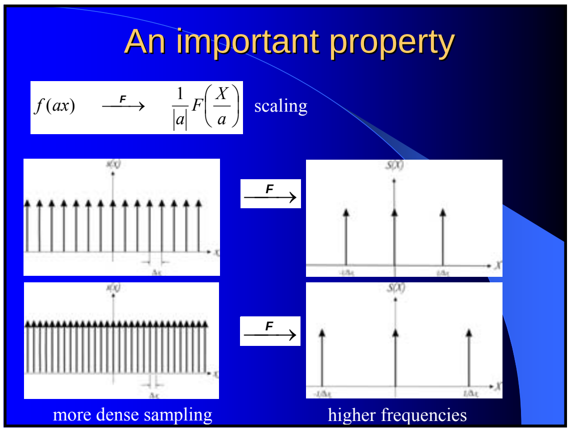

What is sampling?
Δx
Δy

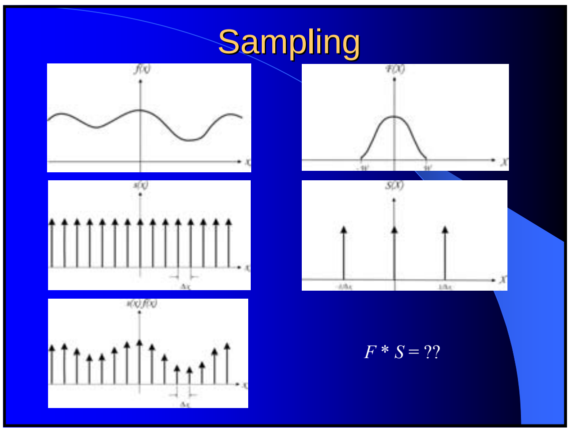

2D sampling
Δx
Δy
sampling lattice
distances: Δx, Δy
uniform sampling

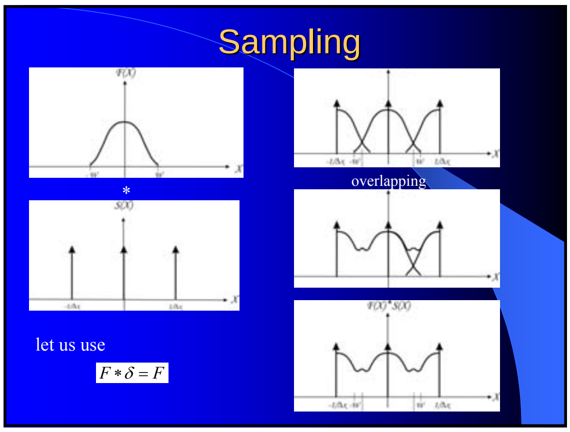

If the sampling is not suitable…
original
4th points
8th points


Impulse function
$\delta(x) \approx \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$
$\delta(x) = \lim_{\alpha \to 0} \alpha^2 e^{-\alpha^2 \pi x^2}$
x
$\delta(x,y) = \delta(x) \cdot \delta(y)$
2D
properties:
$\int_{-\infty}^{\infty} \delta(x) dx = 1,$
$\int_{-\infty}^{\infty} f(x) \cdot \delta(x) dx = f(0),$
$[f * \delta](x) = \int_{-\infty}^{\infty} f(\xi) \cdot \delta(x - \xi) d\xi = f(x),$
$[F\delta](X) = \int_{-\infty}^{\infty} \delta(x) \cdot e^{-2\pi i x X} dx = e^{-2\pi i 0 X} = 1$

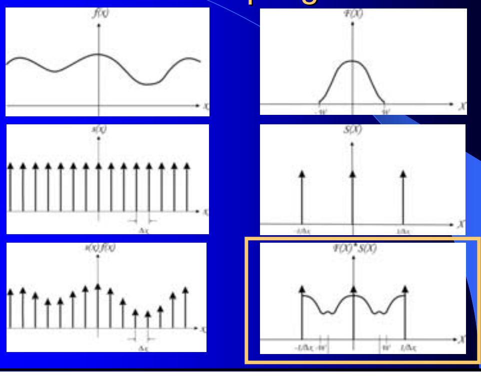

Sampling function
$s(x) = \sum_{j=-\infty}^{\infty} \delta(x - j \cdot \Delta x)$
properties:
F


An important property
$f(ax) \xrightarrow{F} \frac{1}{|a|} F\left(\frac{X}{a}\right)$
scaling
F
F
more dense sampling
higher frequencies

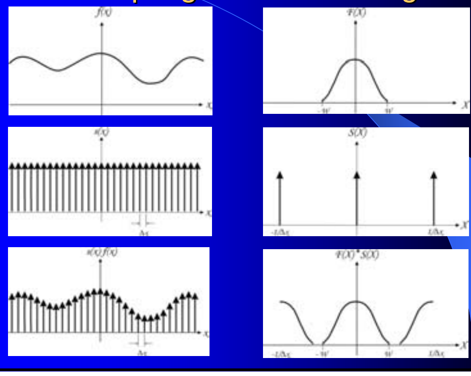

Sampling
$F * S = ??$


Sampling
*
overlapping
let us use
$F * \delta = F$

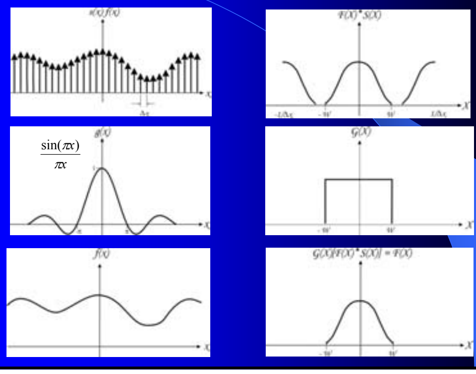
Sampling



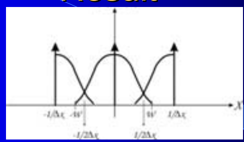
If the sampling is dense enough …



… then an ILPF …



Result



If the function f has no arbitrary big frequencies
(i.e., its Fourier transformation F has bounded support, say, there is a W such that $F(X) = 0$ if $|X| > W$)
and
the sampling of f is dense enough
(i.e., $\Delta x \leq 1/(2 \cdot W)$ or, equivalently, $W \leq 1/(2 \cdot \Delta x)$)

then

f can be reconstructed from its discrete samples.

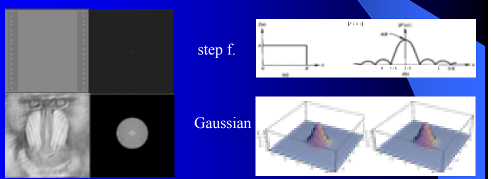
Band limited functions

$f(x)$ is *band limited* if there is a W such that
$F(X)=0$ if $|X|>W$
that is, f has no bigger frequency than W

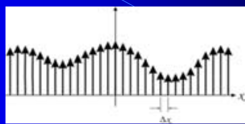
examples: counterexamples:

sin, cos — step f.

ILPF — Gaussian



Whittaker-Shannon theorem



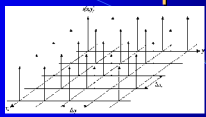
Let $f(x)$ be a band-limited signal with $F(X) = 0$ for $|X|>W$. Then $f(x)$ is uniquely determined by its samples $f(n/(2W))$, $n = 0, \pm 1, \pm 2, \ldots$,

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \cdot \frac{\sin[\pi(2Wx - n)]}{\pi(2Wx - n)}$$

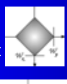
Nyquist frequency: $1/(2 \cdot \Delta x)$

2D sampling

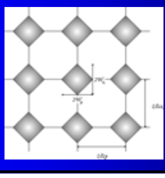
2D sampling function:



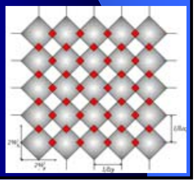
support of the 2D Fourier transformation:



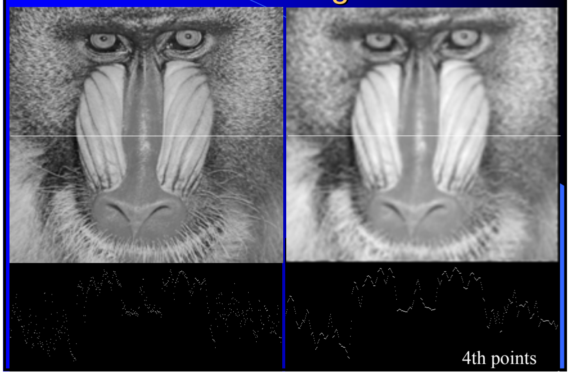
if the sampling is dense enough:



if not



Aliasing



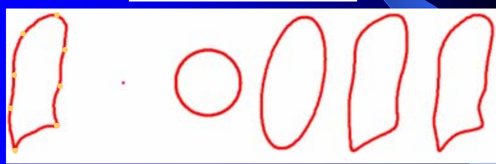
Shape description and representation

Planar shape representation

$f_k = x_k + i \cdot y_k$, $k=0,1,\dots,N-1$, points on the complex plane

$$f_0, f_1, \dots, f_{N-1} \xrightarrow{F} F_0, F_1, \dots, F_{N-1}$$

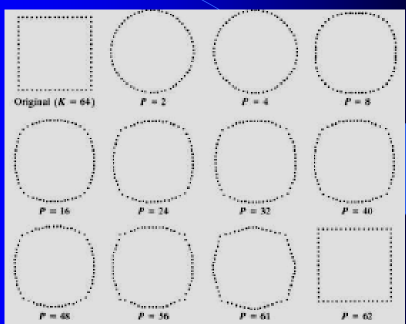
$$f_k = \frac{1}{N}\sum_{l=0}^{N-1} F_l e^{i\frac{2\pi}{N}kl}, k = 0,1,\dots,N-1$$



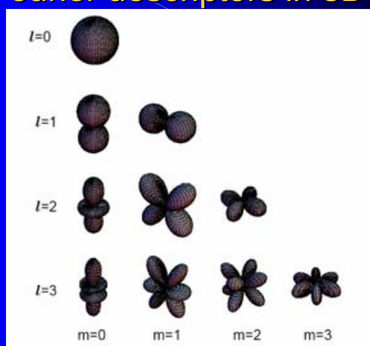
original 1 2 3 10 20

number of harmonics
(i.e., Fourier coefficients/descriptors)

Planar shape representation

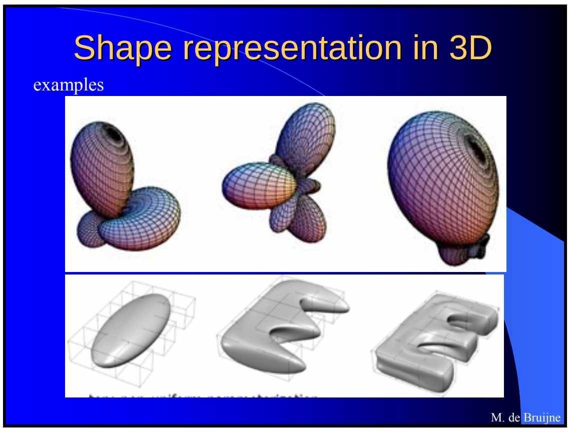


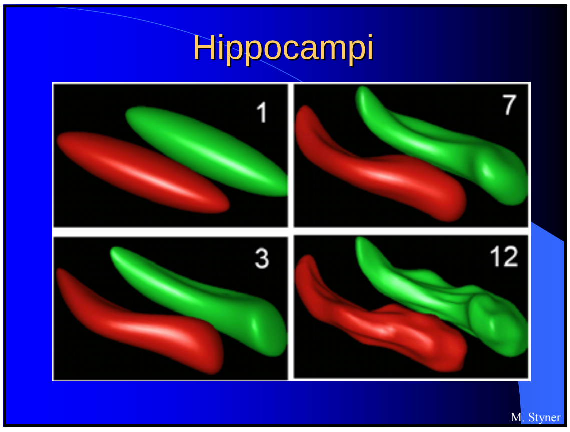
Fourier descriptors in 3D



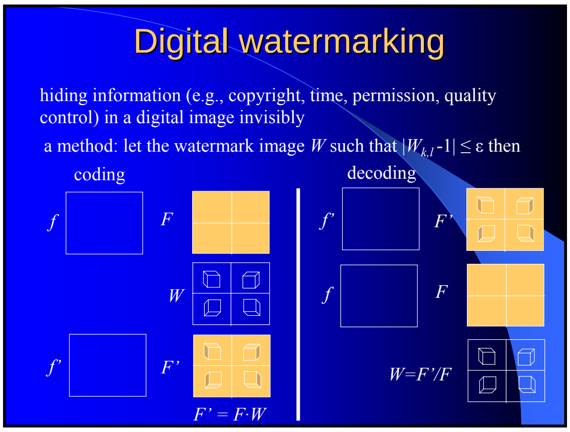
basis functions: spherical harmonics

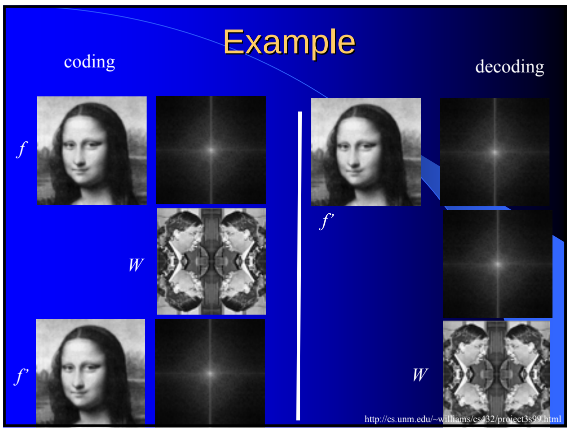
M. de Bruijne


Shape representation in 3D
examples
M. de Bruijne


Hippocampi
1
7
3
12
M. Styner


Digital watermarking


Digital watermarking
hiding information (e.g., copyright, time, permission, quality control) in a digital image invisibly
a method: let the watermark image W such that |W_{k,l}-1| ≤ ε then
coding
decoding
f
F
W
f'
F'
F' = F·W
f'
F'
f
F
W=F'/F


Example
coding
decoding
f
W
f'
f'
W
http://cs.unm.edu/~williams/cs432/project3s99.html