Cross-spectral analysis of signal improvement by stochastic resonance in bistable systems

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ABSTRACT

We consider two bistable systems, the double-well potential and the Schmitt-trigger, and examine whether the stochastic resonance occurring in these systems may produce output signals less noisy than the input. We apply cross-spectrum and cross-correlation based generalised measures to quantify noise content in the input and output, which enables us to use aperiodic or random sequences as input signals. We show that input-output signal improvement occurs even for these types of input.

1. INTRODUCTION

The term 'stochastic resonance' (SR) denotes the ability of some non-linear systems to reduce the noise content of their output through the addition of some noise to the input. SR, in itself, means that the output is less noisy with a certain amount of noise at the input than without noise; yet it is also a valid question whether the same phenomenon may also entail an input-output improvement. As the most common measure of the noise content of a signal is the signal-to-noise ratio (SNR), this input-output improvement is mainly referred to as SNR gain. The study of SNR gain is gaining impetus from the potential of applicability in biological systems, information theory or even in technical arrangements (eg, as filters), and has established itself as a major area within noise research. SNR gains much greater than unity have been found in a wide range of systems showing SR, from the level-crossing detector through the Schmitt trigger to the archetypal double-well model or even neuron models.

These studies show that one can expect significant SNR gains for pulse-like signals; it is highly questionable, however, whether the conventional SNR is a valid measure of noise content in these cases, since here the signal power is not localised at the fundamental frequency yet SNR takes only the first harmonic into account, thus yielding a less than faithful picture. One can circumvent this problem in a number of ways: apply a wide-band extension of SNR, borrow the concept of mutual information from information theory or use correlation analysis.

Here we opt for the third path and reach back to the first study that showed significant input-output improvement by SR[1] and adopt a cross-spectral measure introduced there, along with a cross-correlation-based quantity suggested by Collins *&al* for aperiodic SR.[2] We shall revisit the bistable systems wherein we have already obtained high SNR gains—the double well[3] and the Schmitt trigger[4]—and examine whether a cross-spectral measure reflects the same signal improvement. Applying these generalised measures makes it possible for us to examine the potential of input-output improvement also in the case of aperiodic, even completely random, input signals.

2. MEASURES OF NOISE CONTENT

To quantify signal improvement by stochastic resonance, one needs a measure which reflects the noise content of a noisy signal. The measure most prevalent in the literature of SR is the *signal-to-noise ratio*, the ratio of signal power and the background noise power spectral density at the fundamental frequency f_0:

$$SNR := \frac{\lim_{\Delta f \to 0} \int_{f_0-\Delta f}^{f_0+\Delta f} S(f)df}{S_N(f_0)}, \tag{1}$$

wherein $S(f)$ denotes the power spectral density (PSD) of the signal and $S_N(f)$ stands for the background noise PSD. So as to keep it distinct from the generalised measures we shall shortly introduce, we shall refer to this definition as the *classical* or *narrow-band SNR* in the following. Note that it is not a simple ratio, since its dimension is Hz.

Two facts limit the range of validity of using this definition: first, it relies on the existence of a fundamental frequency, and thus its use is out of the question for aperiodic signals; second, even for periodic but wide-band signals (eg, a periodic pulse train), it does not take into account the signal power at higher harmonics, distorting the power ratio of signal and noise. So, a wide range of input signals calls for generalised measures which circumvent these problems.

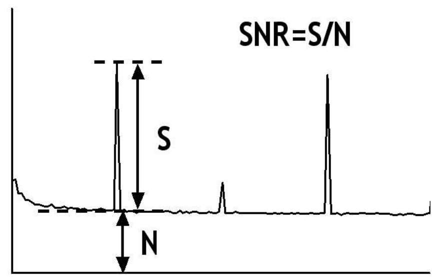

Figure 1. A heuristic method of separating signal from noise.

The chief difficulty lies in separating signal from noise, especially at the output. In the case of the classical definition, output noise is often obtained as simply the output at zero input signal amplitude. This cannot be expected to yield an adequate result when the system operates in the strongly non-linear range and there are cross-modulation products. Another implementation of the same classical SNR (which we adopted) takes the noisy signals as a whole and relying on the condition of smooth noise spectrum, calculates the noise background as the average noise level around the signal peak, and the signal PSD is the total PSD minus this noise background (see Fig. 1). Neither of these is applicable in the case of aperiodic signals (there is no localised signal peak), which led L B Kish to introduce a more robust method of signal-noise separation based on cross-spectra of the pure input signal and the noisy output signal. In this framework, the signal component in the noisy output is the part which shows correlation to the input[1]:

$$S_{out}^{sig}(f) = \frac{|S_{in,\ out}(f)|^2}{S_{in}^{sig}(f)}, \tag{2}$$

where $S_{in,\ out}(f)$ denotes the cross power spectral density of the input signal and the *total* output, while $S_{in}^{sig}(f)$ is the PSD of the input signal. The noise component follows naturally:

$$S_{out}^{noi}(f) = S_{out}^{tot}(f) - S_{out}^{sig}(f), \tag{3}$$

where $S_{out}^{tot}(f)$ is the PSD of the total output. These separated quantities can later be used to calculate the signal-to-noise ratio. A similar correlation-based treatment was used by Collins *&al* to characterise aperiodic SR,[2] only in time domain. Their *power norm* was a normalised cross-correlation coefficient of the input and the output. This time-domain approach has the advantage that it also provides information, apart from the noise content of the signals, about the input-otput time lag.

For the periodic but wide-band signals we used in our previous studies, we suggested a wide-band extension of the classical SNR which takes into account all signal harmonics and the total power of noise.[3,4] This wide-band SNR separates signal from noise the same way as described above (namely, by subtracting background noise averages from signal peaks in the total spectrum), therefore it is not applicable in the case of aperiodic input signals. For periodic signals, it is expected to yield the same results as the cross-spectrum-based measure.

To sum it up, we shall use the following measures in this paper:

$$SNR_w := \frac{\sum_{k=1}^{\infty} \lim_{\Delta f \to 0} \int_{kf_0-\Delta f}^{kf_0+\Delta f} S(f)df}{\int_0^{\infty} S_N\left(f\right) df}, \tag{4}$$

$$SNR_{cs,\,out} := \frac{\int_0^{\infty} S_{out}^{sig}\left(f\right) df}{\int_0^{\infty} S_{out}^{noi}\left(f\right) df}, \tag{5}$$

$$C_{out}\left(\tau\right) := \frac{E\left(\left[p\left(t\right) - E\left(p\right)\right] \cdot y_{out}\left(t+\tau\right)\right)}{\sqrt{E\left(\left[p\left(t\right) - E\left(p\right)\right]^2\right)} \cdot \sqrt{E\left(\left[y_{out}\left(t+\tau\right)\right)\right]^2\right)}}, \tag{6}$$

wherein $S\left(f\right)$ and $S_N\left(f\right)$ are signal and noise power spectral densities either at the input or at the output (separated from each other with the background averaging-subtracting method depicted in Fig. 1); $S_{out}^{sig}\left(f\right)$ and $S_{out}^{noi}\left(f\right)$ are defined by Eqs. (2) and (3), $p\left(t\right)$ denotes the pure (noiseless) input signal, $y_{out}\left(t\right)$ is the *total* output signal and $E\left(p\right)$ is the expected value of p. This formula is the normalised cross-correlation function of $p\left(t\right)$ and $y_{out}\left(t\right)$ with the assumption that $E\left(y_{out}\right) \equiv 0$ (this assumption becomes relevant in the double-well system with sub-threshold input signals, because in this case the output oscillates around the minimum of the potential and has consequently a non-zero mean, which distorts the cross-correlation function).

Since we are mainly interested in the potential of input-output signal improvement, we now define the input counterparts of the quantities listed above:

$$SNR_{cs,\,in} := \frac{\int_0^{\infty} S_{in}^{sig}\left(f\right) df}{\int_0^{\infty} S_{in}^{noi}\left(f\right) df}, \tag{7}$$

wherein

$$S_{in}^{noi}\left(f\right) = S_{in}^{tot}\left(f\right) - S_{in}^{sig}\left(f\right), \tag{8}$$

and $S_{in}^{tot}\left(f\right)$ denotes the PSD of the *total* input.

$$C_{in}\left(\tau\right) := \frac{E\left(\left[p\left(t\right) - E\left(p\right)\right] \cdot y_{in}\left(t+\tau\right)\right)}{\sqrt{E\left(\left[p\left(t\right) - E\left(p\right)\right]^2\right)} \cdot \sqrt{E\left(\left[y_{in}\left(t+\tau\right)\right)\right]^2\right)}}, \tag{9}$$

where $y_{in}\left(t\right)$ is the *total* input in time domain. The quantities to reflect the input-output signal improvement are the wide-band and cross-spectral gains, along with a cross-correlation quotient:

$$G_w := \frac{SNR_{w,\,out}}{SNR_{w,\,in}}, \tag{10}$$

$$G_{cs} := \frac{SNR_{cs,\,out}}{SNR_{cs,\,in}}, \tag{11}$$

$$Q := \frac{C_{out,\,max}}{C_{in,\,max}}, \tag{12}$$

wherein $C_{out,\,max}$ and $C_{in,\,max}$ are the maxima of the output and input cross-correlation functions defined by Eqs. (6) and (9), respectively (the location of the maxima provides information about the input-output lag). In what follows, we shall focus on the behaviour of these quantities in two bistable systems, the double-well potential and the Schmitt trigger, applying both periodic and aperiodic input excitations.

3. INPUT-OUTPUT IMPROVEMENT IN THE DOUBLE-WELL SYSTEM

The so-called double-well potential first emerged in the context of ice age cycles,[5] and has since become an archetypal dynamical model in the study of SR. We consider the overdamped motion of a particle in this double-well potential; the position $x(t)$ of the particle is given by the following Langevin equation:

$$\frac{dx}{dt} = x(t) - x^3(t) + p(t) + w(t), \tag{13}$$

wherein $p(t)$ denotes the input signal and $w(t)$ stands for the noise (Gaussian white noise in our case). We compare the input to the output and look for an input-output improvement through stochastic resonance. This improvement strongly depends on the type of the input signal: for a sinusoidal input, one can expect SNR gains above one only for strongly supra-threshold signals,[6] while a pulse-like input can yield significant gains well below the system threshold.[3] The gains in the latter case increase if the pulses are narrower. Here we supplement our previous studies with the use of further characteristic measures (G_{cs} and Q) and aperiodic input signals.

We used a mixed-signal simulation system to represent the double-well potential as it is faster and not sensitive to numerical artefacts. We generated the input signal and the noise digitally, then applied D/A converters to translate these into analogue signals. To solve Eq. (13) with analogue components, we first first transformed it into an integral form:

$$x(t) = \int_0^t \left\{ x(\tau) - x^3(\tau) + p(\tau) + w(\tau) \right\} d\tau. \tag{14}$$

We realised all mathematical operations in this integral equation using analogue circuits. The output of this circuitry was the solution of Eq. (14), which we then transmitted through an anti-aliasing filter and converted back to the digital domain using high-resolution A/D converters. The whole system was driven by a high-performance digital signal processor (DSP). A host personal computer running LabVIEW controlled the DSP and performed all evaluation tasks. Fig. 2 shows the block diagram of the simulation system.

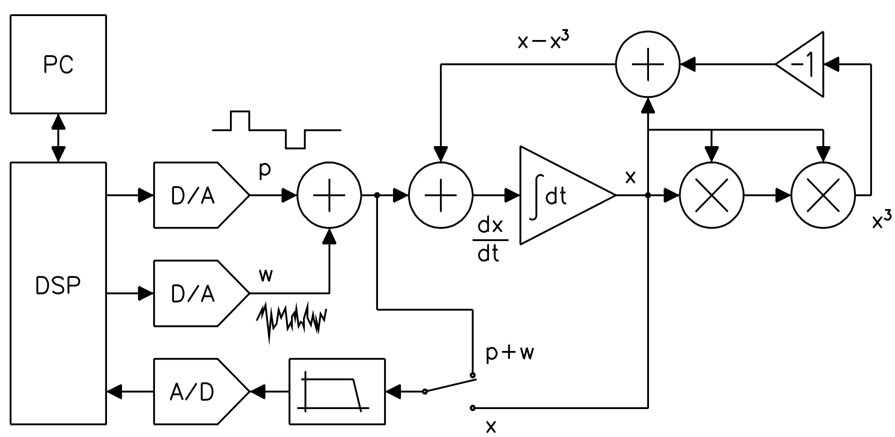

Figure 2. Our mixed-signal simulation system.

It is important to note that the analogue components (namely, the integrator) introduce a $1/t_0$ factor into the right side of Eq. (14), and this modifies the original Langevin equation in the following way:

$$t_0 \cdot \frac{dx}{dt} = x(t) - x^3(t) + p(t) + w(t), \tag{15}$$

The value of t_0 was $1.2 \cdot 10^{-4}$ s; one must re-scale all frequency values with this value if one is to use the original Langevin equation given by Eq. (13).

We used three different types of input signals: the periodic pulse train for which we had already obtained high SNR gains,[3] an aperiodic version of it in which the location of the pulses is somewhat randomised and a noise with low cut-off frequency (see Fig. 3). For the pulse trains, we defined the *duty cycle* of the pulse as $2\tau/T$, where τ is the pulse width and T is the period of the periodic pulse train. For periodic inputs, G_w, G_{cs} and Q are all applicable; in an aperiodic case, only G_{cs} and Q are valid measures.

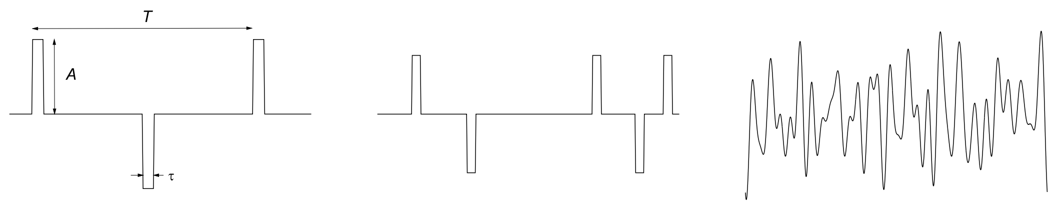

Figure 3. The shape of the input signals: periodic pulse train, aperiodic pulse train and a band-limited noise.

We carried out all our measurements with the following parameters:

- length of samples: 8192;
- number of cycles per sample: 32;
- sampling frequency: 10 kHz;
- distance between pulses (aperiodic pulse train): between 3.2 ms (32 data points) and 22.4 ms (224 data points);
- pulse width (periodic and aperiodic pulse trains): 1.3 ms (13 data point);
- duty cycle (periodic pulse train): 10%;
- bandwidth of noise as input signal: 39 Hz (this pertains to Eq. (15) and corresponds to $4.68 \cdot 10^{-3}$ Hz in Eq. (13));
- number of averages per data sequence: between 10 and 50 (depending on how divergent the data were).

We determined the threshold amplitude A_T experimentally as the minimum signal amplitude at which switching between wells can occur without noise, and expressed all signal amplitudes as a percentage of this threshold.

The results of our measurements are displayed in Figs. 4—7. Fig. 4 depicts the SNR gains G_w and G_{cs} and the correlation ratio Q as functions of noise strength, for a periodic pulse train with an amplitude of 90%. Fig. 5 illustrates the same for a signal of 50% amplitude. We can see that in the case of this periodic input signal, the wide-band gain G_w and the cross-spectral gain G_{cs} yield essentially the same results (as theory suggests), but the input-output improvement is only moderately reflected in the correlation ratio Q because the shapes of the input and output signals are different, and time-domain measures are more sensitive to this difference than spectral ones. The improvement is considerable for close-to-threshold signals, but the output-input ratios exceed 1 even for weak signals (50% of threshold amplitude). The latter seems to contradict the corollary

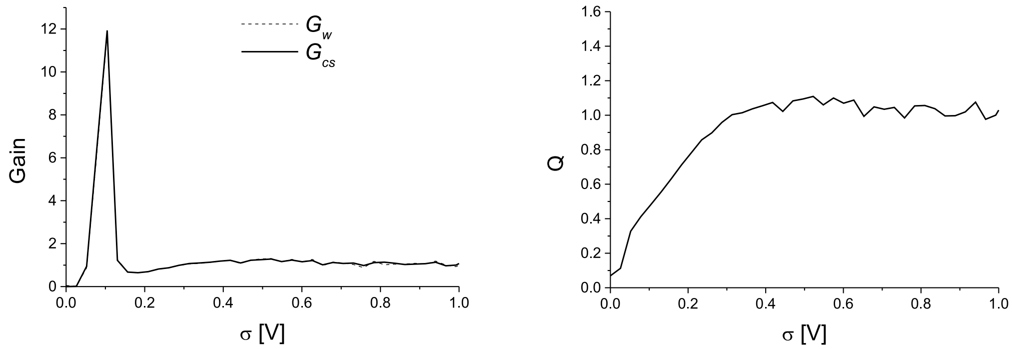

Figure 4. SNR gains (*left panel*) and the cross-correlation ratio (*right panel*) in the double well for a periodic pulse train. The amplitude of the signal is 90% of the threshold and the duty cycle is 10%. The standard deviation of the noise is denoted by σ.

of linear response theory (LRT) which ruled out the possibility of SNR gain in the linear-response regime[7] (that is, for weak signals and strong noise, under which conditions signal transfer through the system can be approximated as linear), but one should bear in mind that this corollary stands only for the classical SNR definition and is not valid for other measures.

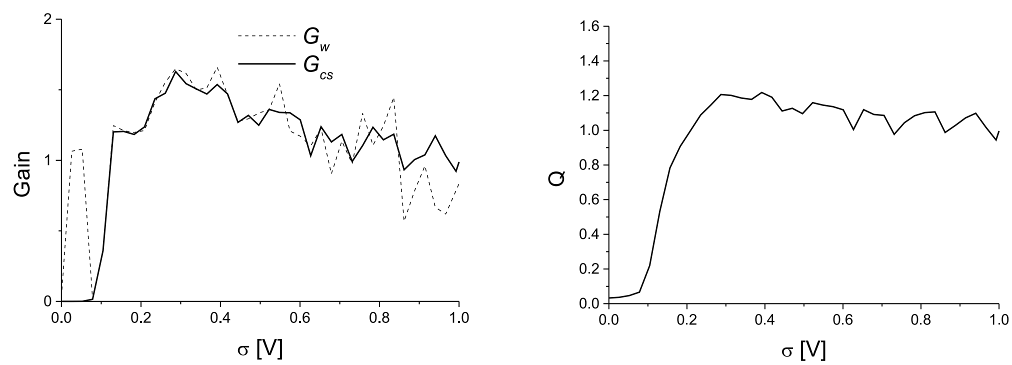

Figure 5. SNR gains (*left panel*) and the cross-correlation ratio (*right panel*) in the double well for a periodic pulse train. The amplitude of the signal is 50% of the threshold and the duty cycle is 10%. The standard deviation of the noise is denoted by σ.

The output-input improvement measures pertaining to the aperiodic, randomised pulse train are illustrated in Fig. 6, while Fig. 7 shows the same for a band-limited noise as input signal. We can observe that even for these aperiodic signals, the stochastic resonance occurring in the double-well system can considerably improve the output compared to the input. The behaviour of G_{cs} in the case of the aperiodic, randomised pulse train is very similar to that in the case of its periodic counterpart, only the maximum of the gain is somewhat smaller.

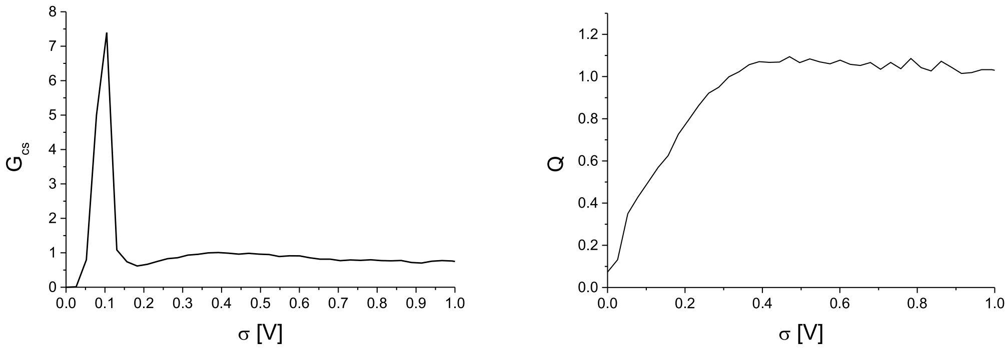

Figure 6. The cross-spectral SNR gain (*left panel*) and the cross-correlation ratio (*right panel*) in the double well for a randomised pulse train. The amplitude of the signal is 90% of the threshold and the pulse width is the same as in the periodic case (1.3 ms). The standard deviation of the noise is denoted by σ.

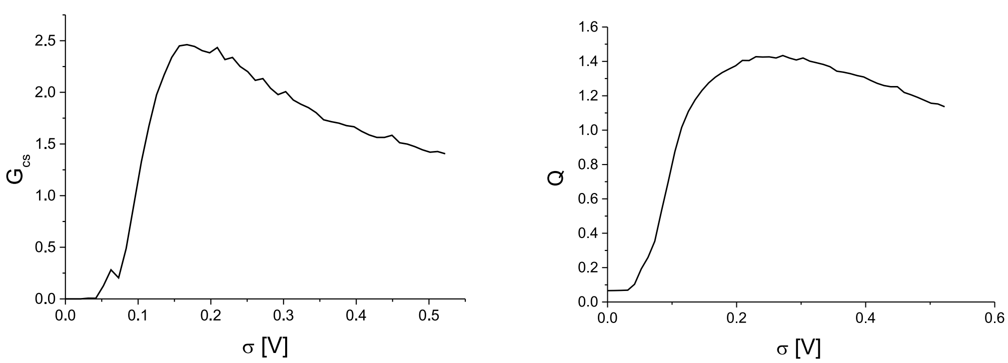

Figure 7. The cross-spectral SNR gain (*left panel*) and the cross-correlation ratio (*right panel*) in the double well for a band-limited noise as input signal. The standard deviation of this noise is 31% of the threshold and its bandwidth is 39 Hz. The standard deviation of the additive noise is denoted by σ.

4. INPUT-OUTPUT IMPROVEMENT IN THE SCHMITT TRIGGER

The other bistable system we study in this work is a simple non-dynamical system, the Schmitt trigger. It is one of the first systems in which stochastic resonance was observed,[8] and later it also proved to be able to produce SNR gain.[4] It has two threshold levels, and its output is dichotomous with an upper and a lower stage. When the input drops below the lower threshold, the output assumes the lower stage and remains so until the input exceeds the upper threshold, when it switches to the upper stage. The output $x(t)$ for a $p(t)$ input signal plus a $w(t)$ Gaussian white noise can be defined in the following way (the thresholds were symmetric, A_T and $-A_T$):

$$x(t) = \begin{cases} 1, & \text{if } p(t) + w(t) > A_T; \\ -1, & \text{if } p(t) + w(t) < -A_T; \\ \text{previous value}, & \text{otherwise.} \end{cases} \tag{16}$$

We carried out numerical simulations in this system, applying a periodic and an aperiodic pulse train as input signals. The parameters were the following:

- length of samples: 32768;
- number of cycles: 32;
- number of averages: 10;
- duty cycle (for periodic pulse) or average duty cycle (for aperiodic pulse): 10%.

The results of our simulations can be seen in Figs. 8—10. We can observe that the Schmitt trigger behaves very similarly to the double well, only here greater gains are obtainable and the divergence between G_w and G_{cs} is more substantial. We get considerable gains also for weak or aperiodic signals.

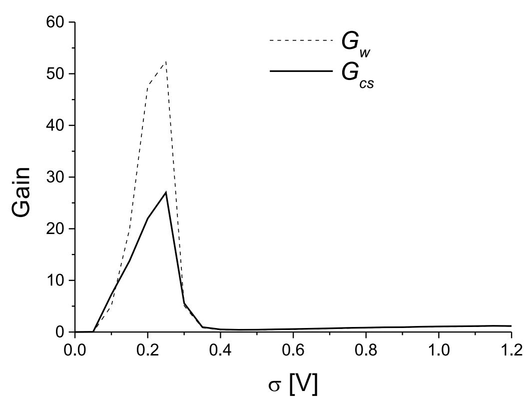

Figure 8. SNR gains in the Schmitt trigger for a periodic pulse train. The amplitude of the signal is 90% of the threshold and the duty cycle is 10%. The standard deviation of the noise is denoted by σ.

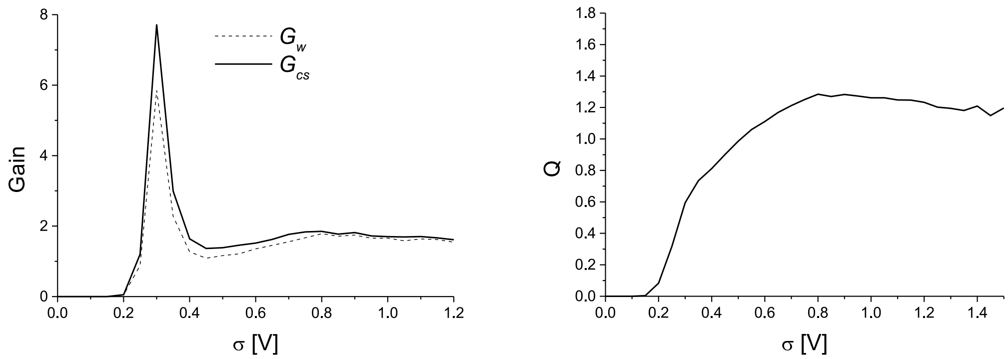

Figure 9. SNR gains (*left panel*) and the cross-correlation ratio (*right panel*) in the Schmitt trigger for a periodic pulse train. The amplitude of the signal is 50% of the threshold and the duty cycle is 10%. The standard deviation of the noise is denoted by σ.

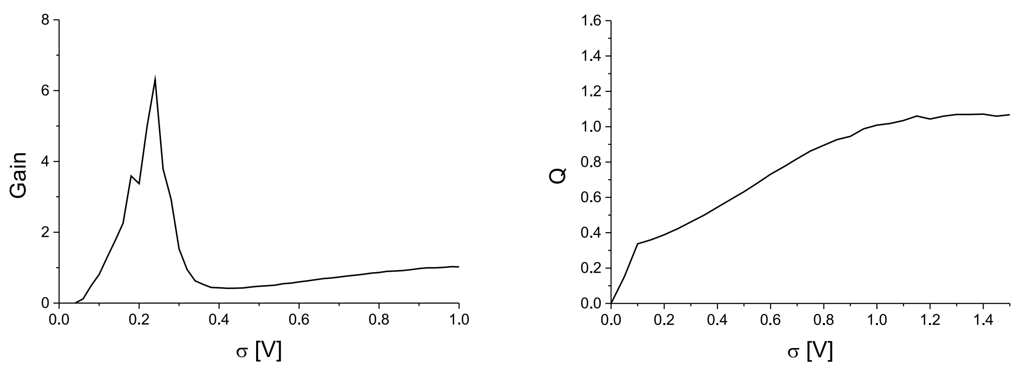

Figure 10. The cross-spectral SNR gain (*left panel*) and the cross-correlation ratio (*right panel*) in the Schmitt trigger for a randomised pulse train. The amplitude of the signal is 90% of the threshold and the average duty cycle is 10%. The standard deviation of the noise is denoted by σ.

5. CONCLUSION

We have considered two bistable systems which had already been reported to show input-output signal improvement through stochastic resonance, the double-well potential and the Schmitt trigger, and supplemented previous studies with the use of cross-spectrum and cross-correlation based generalised measures. These measures made it possible for us to apply aperiodic input signals, for which the classical SNR is not an appropriate measure.

Our mixed-signal measurements and numerical simulations have confirmed that for periodic inputs, the cross-spectrum-based signal-to-noise ratio and the SNR gain derived from it gives essentially the same results (as one would expect from theory) as the wide-band SNR and gain we had already used in our papers, while the former is also applicable for aperiodic types of input. More interestingly, our results have widened the range of input signals for which bistable systems have been shown to exhibit input-output signal improvement: we have given proof that significant SNR gains can be obtained even for weak or aperiodic signals (including a completely random noise as input signal). This suggests that the strong requirements reported in previous studies that the input signal should meet in order to yield high SNR gains (namely, that it should-be pulse-like and close to the system threshold) follow from the classical SNR definition rather than from the mechanism of SR itself.

Throughout this study, we applied Gaussian white noise as the stochastic excitation, yet $1/f^{\alpha}$-type coloured noises are not at all without relevance as they may play a special role in biological systems. Our mixed-signal simulation environment can be supplemented with a digital filter arrangement which would enable us to generate almost arbitrary $1/f^{\alpha}$-type coloured noises in the analogue domain; however, the incorporation of these noises into the class of applied stochastic excitations remains to be the task of future investigations.

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REFERENCES

1. L. B. Kiss, “Possible breakthrough: significant improvement of signal-to-noise ratio by stochastic resonance,” in *Chaotic, Fractal and Nonlinear Signal Processing*, R. Katz, ed., **375**, pp. 382–396, 1996.
2. J. J. Collins, C. C. Chow, and T. T. Imhoff, “Aperiodic stochastic resonance in excitable systems,” *Physical Review E* **52**, pp. R3321–R3324, 1995.
3. Z. Gingl, P. Makra, and R. Vajtai, “High signal-to-noise ratio gain by stochastic resonance in a double well,” *Fluctuation and Noise Letters* **1**, pp. L181–L188, 2001.
4. Z. Gingl, R. Vajtai, and L. B. Kiss, “Signal-to-noise ratio gain by stochastic resonance in a bistable system,” *Chaos, Solitons and Fractals* **11**, pp. 1929–1932, 2000.
5. R. Benzi, A. Sutera, and A. Vulpiani, “The mechanism of stochastic resonance,” *Journal of Physics A* **14**, pp. L453–L457, 1981.
6. P. Hänggi, M. E. Inchiosa, D. Fogliatti, and A. R. Bulsara, “Nonlinear stochastic resonance: The saga of anomalous output-input gain,” *Physical Review E* **62**, pp. 6155–6163, 2000.
7. M. I. Dykman, D. G. Luchinsky, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, “Stochastic resonance in perspective,” in *Fluctuations in Physics and Biology: Stochastic Resonance, Signal Processing and Related Phenomena*, A. Bulsara, ed., *Il Nuovo Cimento* **17D**, pp. 661–683, 1995.
8. S. Fauve and F. Heslot, “Stochastic resonance in a bistable system,” *Physics Letters A* **97**, pp. 5–7, 1983.