Myhill-Nerode theory for fuzzy languages and automata

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Fuzzy Sets

The original Zadeh's definition of a fuzzy set is:

A fuzzy subset of a set A is any mapping $f : A \longrightarrow [0, 1]$, where $[0, 1]$ is the real unit closed interval.

For $x \in A$, the value $f(x)$ is interpreted as

the degree of membership of x to f

that is

the truth value of the proposition '$x \in f$'

Of course, if f takes values only in the set $\{0, 1\}$, then it is treated as an ordinary crisp subset of A.

Structures of Truth Values

Nowadays, various more general structures of truth values are used instead of $[0,1]$.

- Gödel algebras (algebraic counterpart of the Gödel logic)
- MV-algebras or Wajsberg algebras (Łukasiewicz logic)
- Product algebras (Product logic)
- BL-algebras (Basic fuzzy logic)
- Heyting algebras (Intuitionistic logic)
- Complete residuated lattices (Residuated logic)
- Complete orthomodular lattices (Quantum logic), and others.

Here we work with complete residuated lattices, which include the first five kinds of the above mentioned algebras as special cases.

Complete Residuated Lattices

A **residuated lattice** is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ satisfying the following conditions

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,

(L3) $\otimes$ and $\to$ form an **adjoint pair**, i.e., they satisfy the **adjunction property**: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \to z.$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{L}$ is called a **complete residuated lattice**.

Complete Residuated Lattices

The operation $\otimes$ is called a **multiplication**, and $\longrightarrow$ a **residuum**.

They are intended for modeling the **conjunction** and **implication** of the corresponding logical calculus.

Supremum $\bigvee$ and infimum $\bigwedge$ are intended for modeling of the **general** and **existential quantifier**, respectively.

A **biresiduum** or **biimplication** in $\mathscr{L}$ is an operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \longrightarrow y) \wedge (y \longrightarrow x),$$

It is used for modeling the **equivalence** of truth values.

A **negation** in $\mathscr{L}$ is a unary operation $\neg$ defined by

$$\neg x = x \longrightarrow 0.$$

Łukasiewicz, Product and Gödel Operations

The most studied and applied set of truth values is the real unit interval $[0,1]$ with

$$x \wedge y = \min(x,y), \qquad x \vee y = \max(x,y),$$

and three important pairs of adjoint operations:

- **Łukasiewicz** operations

$$x \otimes y = \max(x+y-1,0), \qquad x \to y = \min(1-x+y,1),$$
$$x \leftrightarrow y = 1-|x-y|, \qquad \neg x = 1-x;$$

- **Product operations**

$$x \otimes y = x \cdot y, \qquad x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ y/x & \text{otherwise} \end{cases},$$

$$x \leftrightarrow y = \frac{\min(x,y)}{\max(x,y)}, \qquad \neg x = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0 \end{cases};$$

- **Gödel operations**

$$x \otimes y = \min(x,y), \qquad x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases},$$

$$x \leftrightarrow y = \begin{cases} 1 & \text{for } x = y \\ \min(x,y) & \text{otherwise} \end{cases}, \qquad \neg x = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0 \end{cases};$$

Another important set of truth values is

$$\{a_0, a_1, \ldots, a_n\}, \quad 0 = a_0 < \cdots < a_n = 1,$$

with

$$a_k \otimes a_l = a_{\max(k+l-n,0)}, \qquad a_k \to a_l = a_{\min(n-k+l,n)}.$$

A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$.

The only adjoint pair on it consist of the classical conjunction and implication operations.

Fuzzy Sets Revisited

Let $\mathscr{L}$ will be a complete residuated lattice.

A fuzzy subset of a set A over $\mathscr{L}$, or simply a fuzzy subset of A, is any mapping $f : A \longrightarrow L$.

The set $\mathscr{F}(A)$ of all fuzzy subsets of A we call the fuzzy power set of A.

For $f, g \in \mathscr{F}(A)$ we define

Equality: $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$

Inclusion: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$

The meet or intersection $\bigwedge_{i\in I} f_i$ and the join or union $\bigvee_{i\in I} f_i$ of a family $\{f_i\}_{i\in I} \subseteq \mathscr{F}(A)$ are mappings from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x), \qquad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x).$$

Fuzzy Sets Revisited

The **crisp part** of a fuzzy subset $f \in \mathscr{F}(A)$ is a crisp set defined by

$$\widehat{f} = \{x \in A \mid f(x) = 1\}.$$

A **fuzzy relation** on A is any mapping $\mu : A \times A \longrightarrow L$, i.e., any fuzzy subset of $A \times A$.

Hence, the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets.

Fuzzy Equivalence Relations

A fuzzy relation μ on A is said to be

(R) **reflexive** or **fuzzy reflexive** if $\mu(x, x) = 1$, for every $x \in A$;

(S) **symmetric** or **fuzzy symmetric** if $\mu(x, y) = \mu(y, x)$, for all $x, y \in A$;

(T) **transitive** or **fuzzy transitive** if for all $x, a, y \in A$

$$\mu(x, a) \otimes \mu(a, y) \leqslant \mu(x, y).$$

A reflexive, symmetric and transitive fuzzy relation on A is called a **fuzzy equivalence relation**, or just a **fuzzy equivalence**, on A.

With respect to the ordering of fuzzy relations, the set $\mathscr{E}(A)$ of all fuzzy equivalence relations on a set A is a complete lattice.

Fuzzy Equivalence Relations

Let μ be a fuzzy equivalence relation on A.

For each $a \in A$ we define $\mu_a \in \mathscr{F}(A)$, i.e., $\mu_a : A \longrightarrow L$, by:

$$\mu_a(x) = \mu(a, x), \quad \text{for every } x \in A.$$

We call μ_a a fuzzy equivalence class, or just an equivalence class, of μ determined by the element a.

The set $A/\mu = \{\mu_a \,|\, a \in A\}$ is called the factor set of A w.r.t. μ. Its cardinality $|A/\mu|$ is called the index of μ, in notation $\text{ind}(\mu)$.

A fuzzy subset $f \in \mathscr{F}(A)$ is said to be extensional w.r.t. μ if

$$f(x) \otimes \mu(x, y) \leqslant f(y),$$

for all $x, y \in A$.

A **fuzzy automaton over $\mathscr{L}$**, or simply a **fuzzy automaton**, is a triple $\mathscr{A} = (A, X, \delta)$, where

- A and X are sets, called respectively a **set of states** and an **input alphabet**,
- $\delta : A \times X \times A \to L$ is a fuzzy subset of $A \times X \times A$, called a **fuzzy transition function**.

We will always assume that the input alphabet X is finite, but from methodological reasons we will allow the set of states A to be infinite.

A fuzzy automaton whose set of states is finite is called a **finite fuzzy automaton**.

Let X^* denote the free monoid over the alphabet X.

The mapping δ can be extended up to a mapping $\delta^* : A \times X^* \times A \to L$ as follows: If $a, b \in A$ and $e \in X^*$ is the empty word, then

$$\delta^*(a, e, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases},$$

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^*(a, ux, b) = \bigvee_{c \in A} \delta^*(a, u, c) \otimes \delta(c, x, b).$$

We have that for all $a, b \in A$ and $u, v \in X^*$,

$$\delta^*(a, uv, b) = \bigvee_{c \in A} \delta^*(a, u, c) \otimes \delta^*(c, v, b).$$

If δ is a crisp subset of $A \times X \times A$, i.e., $\delta : A \times X \times A \longrightarrow \{0, 1\}$, then $\mathscr{A}$ is an ordinary crisp nondeterministic automaton.

Moreover, if δ is a mapping of $A \times X$ into A, then $\mathscr{A}$ is an ordinary deterministic automaton.

Evidently, in these two cases we have that δ^* is also a crisp subset of $A \times X^* \times A$, and a mapping of $A \times X^*$ into A, respectively.

Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton.

Then $\widehat{\delta}$, the crisp part of δ, is a crisp subset of $A \times X \times A$, and $\widehat{\mathscr{A}} = (A, X, \widehat{\delta})$ is a nondeterministic automaton, called the crisp part of the fuzzy automaton $\mathscr{A}$.

A **fuzzy language** is any fuzzy subset of a free monoid.

A fuzzy automaton $\mathscr{A} = (A, X, \delta)$ is said to **recognize** a fuzzy language $f \in \mathscr{F}(X^*)$, by a fuzzy set σ of **initial states** and a fuzzy set τ of **final states**, if for any $u \in X^*$,

$$f(u) = \bigvee_{a,b\in A} \sigma(a) \otimes \delta^*(a, u, b) \otimes \tau(b).$$

Here we consider only those fuzzy automata having a single crisp initial state $\{a_0\}$.

In this case, $\mathscr{A}$ is said to recognize a fuzzy language $f \in \mathscr{F}(X^*)$ by a crisp initial state a_0 and a fuzzy set τ of final states, if for any $u \in X^*$,

$$f(u) = \bigvee_{b\in A} \delta^*(a_0, u, b) \otimes \tau(b).$$

Fuzzy Languages

In particular, if $\mathscr{A}$ is a deterministic automaton, i.e., $\delta : A \times X \longrightarrow A$, then it recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ by a crisp initial state a_0 and a fuzzy set τ of final states, if for any $u \in X^*$,

$$f(u) = \tau(\delta^*(a_0, u)).$$

Fuzzy Right Congruence Automata

A fuzzy equivalence relation μ on a semigroup S is

- a **fuzzy left congruence**, if $\mu(a,b) \leqslant \mu(xa,xb)$, for all $a,b,x \in S$,
- a **fuzzy right congruence**, if $\mu(a,b) \leqslant \mu(ax,bx)$, for all $a,b,x \in S$,
- a **fuzzy congruence**, if it is both fuzzy left and right congruence.

For a fuzzy equivalence relation μ on a semigroup S we define fuzzy relations μ_l^0, μ_r^0 and μ^0 on S by

$$\mu_l^0(a,b) = \bigwedge_{x\in S^1} \mu(xa,xb), \quad \mu_r^0(a,b) = \bigwedge_{x\in S^1} \mu(ax,bx),$$

$$\mu^0(a,b) = \bigwedge_{x,y\in S^1} \mu(xay,xby),$$

for all $a,b \in S$.

We have that

- μ_l^0 is the largest fuzzy left congruence on S contained in μ;
- μ_r^0 is the largest fuzzy right congruence on S contained in μ;
- μ^0 is the largest fuzzy congruence on S contained in μ;
- if μ is a fuzzy right (left) congruence, then $\mu^0 = \mu_l^0$ ($\mu^0 = \mu_r^0$).

Therefore, the mappings $\mu \mapsto \mu_l^0$, $\mu \mapsto \mu_r^0$ and $\mu \mapsto \mu^0$ are opening operators on the lattice of fuzzy equivalence relations on S, so

- μ_l^0 is called the fuzzy left congruence opening of μ,
- μ_r^0 is the fuzzy right congruence opening of μ,
- μ^0 is the fuzzy congruence opening of μ.

Now, we will consider fuzzy right congruences on free monoids.

Let μ be a fuzzy right congruence on a free monoid X^* and let $A_\mu = X^*/\mu$. We define a mapping $\delta_\mu : A_\mu \times X \times A_\mu \longrightarrow L$ by

$$\delta_\mu(\mu_u, x, \mu_v) = \mu_{ux}(v), \tag{1}$$

for all $u, v \in X^*$ and $x \in X$.

The mapping δ_μ is well-defined, and $\mathscr{A}_\mu = (A_\mu, X, \delta_\mu)$ is a fuzzy automaton, called a **fuzzy right congruence automaton** associated with μ.

The transition function can be extended to a function $\delta^*_\mu : A_\mu \times X^* \times A_\mu \longrightarrow L$ by:

$$\delta^*_\mu(\mu_u, p, \mu_v) = \mu_{up}(v) = \mu(up, v), \tag{2}$$

for all $u, v \in X^*$ and $p \in X^+$.

Note that δ_μ^* can be also characterized as follows:

$$\delta_\mu^*(\mu_u, p, \mu_v) = \bigwedge_{w\in X^*} \mu_{up}(w) \leftrightarrow \mu_v(w) = \bigvee_{w\in X^*} \mu_{up}(w) \otimes \mu_v(w),$$

for all $u, v, p \in X^*$.

These equalities can be interpreted as
"$\delta_\mu^*(\mu_u, p, \mu_v)$ is the degree of equality of the classes μ_{up} and μ_v", or
"$\delta_\mu^*(\mu_u, p, \mu_v)$ is the degree of intersection of the classes μ_{up} and μ_v"

A fuzzy right congruence automaton $\mathscr{A}_\mu$ is usually considered as a fuzzy automaton with a crisp initial state μ_e, and then we write

$$\mathscr{A}_\mu = (A_\mu, X, \mu_e, \delta_\mu).$$

When we recognize fuzzy languages by $\mathscr{A}_\mu$ we always assume that $\mathscr{A}_\mu$ starts from the crisp initial state μ_e.

We say that the automaton $\mathscr{A}_\mu$ **recognizes a fuzzy language** $f \in \mathscr{F}(X^*)$ by a fuzzy set of final states $\tau \in \mathscr{F}(A_\mu)$ if

$$f(u) = \bigvee_{\xi \in A_\mu} \delta_\mu^*(\mu_e, u, \xi) \otimes \tau(\xi) = \bigvee_{w \in X^*} \delta_\mu^*(\mu_e, u, \mu_w) \otimes \tau(\mu_w),$$

for each $u \in X^*$.

Our main result is

Theorem 2. Let μ be a fuzzy right congruence on a free monoid X^*. A fuzzy language $f \in \mathscr{F}(X^*)$ is recognized by $\mathscr{A}_\mu$ if and only if f is extensional with respect to μ.

Fuzzy Right Congruence Automata

As known, to any crisp right congruence π on a free monoid X^* we can associate a crisp deterministic automaton $\mathscr{A}_\pi = (A_\pi, X, \lambda_\pi)$, where $A_\pi = X^*/\pi$ and a mapping $\lambda_\pi : A_\pi \times X \longrightarrow A_\pi$ is defined by

$$\lambda_\pi(\pi_u, x) = \pi_{ux}, \tag{3}$$

for all $u \in X^*$ and $x \in X$.

Also, λ_π can be extended up to $\lambda_\pi^* : A_\pi \times X^* \longrightarrow A_\pi$ so that

$$\lambda_\pi^*(\pi_u, v) = \pi_{uv}, \tag{4}$$

for all $u, v \in X^*$.

We prove the following:

Theorem 3. Let μ be a fuzzy right congruence on X^* and let $\widehat{\mu}$ be its crisp part. Then

(a) $\mathscr{A}_{\widehat{\mu}}$ is the crisp part of $\mathscr{A}_\mu$;

(b) any $f \in \mathscr{F}(X^*)$ recognized by $\mathscr{A}_\mu$ is also recognized by $\mathscr{A}_{\widehat{\mu}}$.

Theorem 4. For any fuzzy language $f \in \mathscr{F}(X^*)$ the following is true:

(a) A fuzzy relation ϱ_f on X^* defined by

$$\varrho_f(u,v) = \bigwedge_{w\in X^*} f(uw) \leftrightarrow f(vw), \quad \text{for any } u,v \in X^*,$$

is the greatest fuzzy right congruence on X^* such that f is extensional w.r.t. to it;

(b) $\mathscr{A}_{\widehat{\varrho}_f}$ is a minimal deterministic automaton recognizing f.

For a fuzzy language $f \in \mathscr{F}(X^*)$ and $u \in X^*$, a fuzzy language $f_u \in \mathscr{F}(X^*)$ defined by

$$f_u(v) = f(uv), \quad \text{for each } v \in X^*,$$

is called a derivative or a (right quotient) of f with respect to u.

Let A_f be the set of all derivatives of f, i.e., $A_f = \{f_u \,|\, u \in X^*\}$, and define a mapping $\delta_f : A_f \times X \times A_f \longrightarrow L$ by

$$\delta_f(f_u, x, f_v) = \bigwedge_{w \in X^*} f_{ux}(w) \leftrightarrow f_v(w), \tag{5}$$

for all $u, v \in X^*$ and $x \in X$.

We prove:

Theorem 5. For any $f \in \mathscr{F}(X^*)$, the mapping δ_f is well-defined and $\mathscr{A}_f = (A_f, X, \delta_f)$ is a fuzzy automaton isomorphic to $\mathscr{A}_{\varrho_f}$.

For a fuzzy language $f \in \mathscr{F}(X^*)$, we also define a mapping $\lambda_f : A_f \times X \longrightarrow A_f$ by

$$\lambda_f(f_u, x) = f_{ux}, \tag{6}$$

for any $u \in X^*$ and $x \in X$.

Evidently, λ_f can be extended up to $\lambda_f^* : A_f \times X^* \longrightarrow A_f$ so that

$$\lambda_f^*(f_u, v) = f_{uv}, \tag{7}$$

for all $u, v \in X^*$.

We also prove:

Theorem 6. For any fuzzy language $f \in \mathscr{F}(X^*)$, the mapping λ_f is well-defined and $\mathscr{B} = (A_f, X, \lambda_f)$ is a deterministic automaton isomorphic to $\mathscr{A}_{\widehat{\varrho}_f}$.

Moreover, $\mathscr{B}$ is the crisp part of $\mathscr{A}_f$, that is $\mathscr{B} = \widehat{\mathscr{A}_f}$.

Theorem 7. For any fuzzy language $f \in \mathscr{F}(X^*)$, both $\mathscr{A}_f$ and $\widehat{\mathscr{A}_f}$ recognize f with the crisp initial state f and the fuzzy set of final states $\tau \in \mathscr{F}(A_f)$ defined by

$$\tau(g) = g(e),$$

for any derivative $g \in A_f$.

Given a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ and a state $a \in A$.

A fuzzy relation ϱ_a on the free monoid X^* defined by

$$\varrho_a(u, v) = \bigwedge_{b \in A} \delta^*(a, u, b) \leftrightarrow \delta^*(a, v, b), \tag{8}$$

for $u, v \in X^*$, is called Nerode's fuzzy relation determined by a.

If $\mathscr{A}$ is an initial fuzzy automaton with a crisp initial state a_0, then the fuzzy relation ϱ_{a_0} is denoted by $\varrho_{\mathscr{A}}$ and called a Nerode's fuzzy relation of the fuzzy automaton $\mathscr{A}$.

We prove the following:

Theorem 8. For any state a of a fuzzy automaton $\mathscr{A} = (A, X, \delta)$, the Nerode's fuzzy relation ϱ_a is a fuzzy right congruence on X^*.

Theorem 9. Any fuzzy language $f \in \mathscr{F}(X^*)$ recognized by a fuzzy automaton $\mathscr{A}$ is also recognized by the fuzzy automaton $\mathscr{A}_{\varrho_{\mathscr{A}}}$.

To a fuzzy automaton $\mathscr{A} = (A, X, \delta)$, we also assign a fuzzy relation $\vartheta_{\mathscr{A}}$ on the free monoid X^* defined by

$$\vartheta_{\mathscr{A}}(u,v) = \bigwedge_{a\in A} \varrho_a(u,v) = \bigwedge_{a,b\in A} \delta^*(a,u,b) \leftrightarrow \delta^*(a,v,b), \tag{9}$$

for $u, v \in X^*$, which is called Myhill's fuzzy relation of the fuzzy automaton $\mathscr{A}$.

Theorem 10. For any fuzzy automaton $\mathscr{A} = (A, X, \delta)$, the Myhill's fuzzy relation $\vartheta_{\mathscr{A}}$ is a fuzzy congruence on X^*.

Theorem 11. Let μ be a fuzzy right congruence on X^*. Then

(a) Nerode’s fuzzy right congruence of $\mathscr{A}_\mu$ coincide with μ;

(b) Muhill’s fuzzy congruence of $\mathscr{A}_\mu$ is the fuzzy congruence opening of μ.

Let $\mathscr{A} = (A, X, a_0, \delta)$ be a fuzzy automaton with a crisp initial state a_0.

We denote by $(L_{\mathscr{A}}, \vee, \otimes)$ the subalgebra of the reduct $(L, \vee, \otimes)$ of $\mathscr{L}$ generated by the set $\{\delta(a, x, b) \,|\, a, b \in A,\ x \in X\}$.

For any $u \in X^*$ let a mapping $\Delta_u : A \to L_{\mathscr{A}}$ be defined by

$$\Delta_u(a) = \delta^*(a_0, u, a),$$

for each $a \in A$, let $A_\Delta = \{\Delta_u \,|\, u \in X^*\}$ and let $\lambda_\Delta : A_\Delta \times X \to A_\Delta$ be defined by

$$\lambda_\Delta(\Delta_u, x) = \Delta_{ux},$$

for all $u \in X^*$ and $x \in X$.

We have the following

Theorem 12. Let $\mathscr{A} = (A, X, a_0, \delta)$ be a fuzzy automaton with a crisp initial state a_0. Then

(a) the mapping λ_Δ is well-defined and $\mathscr{A}_\Delta = (A_\Delta, X, \lambda_\Delta)$ is an automaton isomorphic to $\mathscr{A}_{\widehat{\varrho}_{\mathscr{A}}}$;

(b) $\operatorname{ind}(\varrho_{\mathscr{A}}) = \operatorname{ind}(\widehat{\varrho}_{\mathscr{A}}) \leqslant |L_{\mathscr{A}}^{A}|$.

By this we deduce the following:

Theorem 13. The following conditions are equivalent:

(i) The reduct $(L, \vee, \otimes)$ of $\mathscr{L}$ is a locally finite algebra;

(ii) Nerode's fuzzy right congruence of any finite fuzzy automaton over $\mathscr{L}$ has a finite index;

(iii) Myhill's fuzzy congruence of any finite fuzzy automaton $\mathscr{L}$ has a finite index.

As a consequence, a result of Li and Pedrycz (Fuzzy Sets and Systems 156 (2005), 68–92) one obtains, which says that (i) is equivalent to

(iv) Any fuzzy language recognizable by a finite fuzzy automaton, is also recognizable by a finite deterministic automaton (over $\mathscr{L}$).

Finally, the second main result is:

Theorem 14. For a fuzzy language $f \in \mathscr{F}(X^*)$, the following five conditions are equivalent if and only if the algebra $(L, \vee, \otimes)$ is locally finite:

(i) f is a recognizable fuzzy language;

(ii) f is extensional with respect to a fuzzy right congruence of finite index;

(iii) f is extensional with respect to a fuzzy congruence of finite index;

(iv) the syntactic fuzzy right congruence ϱ_f has a finite index;

(v) the syntactic fuzzy congruence ϑ_f has a finite index.

Concluding Remarks

(1) Syntactic right congruences, syntactic congruences and derivatives of fuzzy languages have been considered in

- Shen (Information Sciences 88 (1996), 149-168)
- Malik, Mordeson and Sen (Inform. Sciences 88 (1996), 263-273)
- Mordeson and Malik's book (Chapman & Hall / CRC, 2002)

Here, fuzzy languages were studied in terms of fuzzy right congruences and fuzzy congruences for the first time.

Nerode's fuzzy right congruence and Myhill's fuzzy congruence of a fuzzy automaton are also new concepts.

(2) The concept of extensionality, which play an outstanding role in our research, has important applications in fuzzy control, fuzzy clustering, and other fields.