The profinite approach to decidability questions

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- Recall that a *pseudovariety* (of semigroups) is a class of finite semigroups closed under H, S, P_{fin}.
- The pseudovariety generated by a class $\mathcal{C}$ of finite semigroups is $HSP_{\text{fin}}(\mathcal{C})$.
- Pseudovarieties are often defined by generators, namely by applying natural algebraic operations to members of given pseudovarieties.
- Operators of special interest are

algebraic operation	operator	notation
direct product	*join*	$\mathbf{V} \vee \mathbf{W}$
semidirect/wreath product	*semidirect product*	$\mathbf{V} * \mathbf{W}$
extensions with prescribed idempotent classes	*Mal'cev product*	$\mathbf{V} ⓜ \mathbf{W}$
power semigroup	*power*	$\mathbf{PV}$

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- Say that a pseudovariety **V** is *decidable* if there is an algorithm to effectively test whether a given finite semigroup belongs to **V** (membership problem).

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Given an operator $\mathbf{O}$ *and (decidable) pseudovarieties* $\mathbf{V}_1, \ldots, \mathbf{V}_n$*, determine whether* $\mathbf{O}(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ *is decidable and, in the affirmative case, find efficient algorithms to test the membership problem.*

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- In general, pseudovarieties do not have free objects.
- Yet, in the context of topological semigroups, there are structures which play such a role, namely *relatively free profinite semigroups*.
- A *profinite semigroup* is a residually finite compact semigroup.
- A *pro-*$\mathbf{V}$ semigroup is a residually-$\mathbf{V}$ compact semigroup.
- A *free pro-*$\mathbf{V}$ *semigroup over* X:

$$\begin{array}{ccc} X & \longrightarrow & \hat{F}_X\mathbf{V} \\ & \searrow^{\varphi} & \downarrow \hat{\varphi} \\ & & S \end{array}$$

  where S is an arbitrary pro-$\mathbf{V}$ semigroup
- Each $u \in \hat{F}_X\mathbf{V}$ defines an operation $u_S : S^X \to S$ by $u_S(\varphi) = \hat{\varphi}(u)$.
- *Pseudoidentities*: write $S \models u = v$ if $u_S = v_S$.

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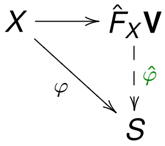

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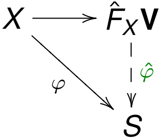

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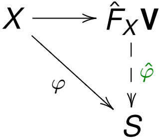

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- $\hat{F}_X\mathbf{V}$ encodes a lot of information about the pseudovariety $\mathbf{V}$:
  - the X-generated members of $\mathbf{V}$ are the finite continuous homomorphic images of $\hat{F}_X\mathbf{V}$;
  - the rational languages $L \subseteq X^+$ whose syntactic semigroups belong to $\mathbf{V}$ are those such that $\overline{\iota(L)}$ is open, where

  - a finite X-generated semigroup S belongs to $\mathbf{V}$ if and only if $S \models u = v$ for all $u, v \in \hat{F}_X\mathbf{S}$ such that $p_{\mathbf{V}}(u) = p_{\mathbf{V}}(v)$;
  - $\therefore$ Reiterman's Theorem: pseudovarieties are defined by pseudoidentities.

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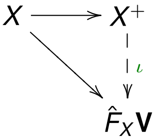

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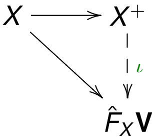

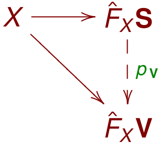

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- Provided pseudoidentities in a basis can be (collectively) effectively checked, the pseudovariety is decidable.

THEOREM (PIN-WEIL'1994)

Suppose that $\mathbf{V} = [\![u_i = v_i : i \in I]\!]$. *Then* $\mathbf{V} ⓜ \mathbf{W}$ *is defined by the pseudoidentities of the form*

$$u_i(w_1, \ldots, w_{n_i}) = v_i(w_1, \ldots, w_{n_i}) \qquad (i \in I)$$

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- Hence decidability follows if we can show $\mathbf{O}(\mathbf{V}, \mathbf{W})$ is co-recursively enumerable.
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- Abstracting from the Mal'cev case, this leads to the following problem for a pseudovariety **W**:
    - given a finite system of equations $U_k = V_k$ ($k = 1, \ldots, m$) in the set X of variables and a continuous homomorphism $\varphi : \hat{F}_X\mathbf{S} \to S$ into a finite semigroup S;
    - we wish to decide whether there exist continuous homomorphisms ψ and δ such that the following diagram commutes

$$\hat{F}_X\mathbf{S} \overset{\psi}{\dashrightarrow} \hat{F}_A\mathbf{S} \xrightarrow{p_{\mathbf{V}}} \hat{F}_A\mathbf{V}, \quad \delta : \hat{F}_A\mathbf{S} \dashrightarrow S, \quad \varphi : \hat{F}_X\mathbf{S} \to S$$

      and $p_{\mathbf{V}}\psi(U_k) = p_{\mathbf{V}}\psi(V_k)$ for $k = 1, \ldots, m$.

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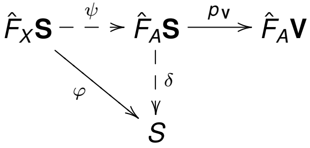

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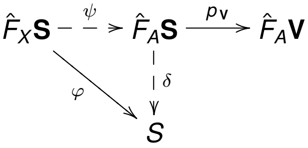

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  - given a finite system of equations $U_k = V_k$ ($k = 1, \dots, m$) in the set X of variables and the choice of a clopen constraint $K_x \subseteq \hat{F}_A\mathbf{S}$ for each $x \in X$;
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- We assume that σ is an implicit signature which has the following properties:
- Since each $w \in \sigma$ has a natural interpretation w_S on each profinite semigroup S, profinite semigroups have a natural structure as σ-algebras.
- Denote by $F_A^\sigma\mathbf{V}$ the σ-subalgebra of $\hat{F}_A\mathbf{V}$ generated by A.
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- Say that **V** is *completely σ-tame* if it is σ-tame with respect to arbitrary finite systems of equations in the signature σ.

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Summary of most important tameness results

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THEOREM (JA-WEIL'1998)

Suppose that $g\mathbf{V} = [\![u_i = v_i; i \in I]\!]$, *where the* $u_i = v_i$ *are semigroupoid pseudoidentities over finite digraphs* Γ_i *with a bounded number of vertices. Then* $\mathbf{V} * \mathbf{W}$ *is defined by the pseudoidentities of the form*

$$z\bar{u}_i = z\bar{v}_i \qquad (i \in I)$$

where the $\bar{u}_i, \bar{v}_i$ *are obtained from* u_i, v_i *by an evaluation of the vertices and edges of* Γ_i, *the initial vertex being assigned the value* z, *which provides a* $\mathbf{W}$*-solution of the system of equations determined by the graph.*

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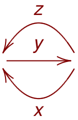
  - since $g\mathbf{Com} = [\![xyz = zyx$ 

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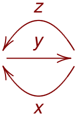
  - since $g\mathbf{Com} = [\![xyz = zyx$ 

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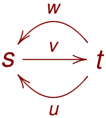

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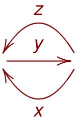
  - since $g\mathbf{Com} = [\![xyz = zyx$ 

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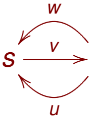

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