MILP: modeling tricks

Alternative sets of constraints

Consider two set of constraints

$$f_i^1(x) \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) \leq b_i^2, i = 1, \ldots, m_2$$

A set of constraints stating that at least one of the two above sets of constraints must be satisfied can be written as

$$f_i^1(x) - \delta_1 M_i^1 \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) - \delta_2 M_i^2 \leq b_i^2, i = 1, \ldots, m_2$$
$$\delta_1 + \delta_2 \leq 1$$
$$\delta_1, \delta_2 \in \{0, 1\}$$

provided that the parameters M_i^j satisfy
$f_i^j(x) \leq b_i^j + M_i^j, i = 1, \ldots, m_j, j = 1, 2$

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$$f_i^2(x) \leq b_i^2, i = 1, \ldots, m_2$$

A set of constraints stating that only one set of contraints must be satisfied can be written as

$$f_i^1(x) - \delta_1 M_i^1 \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) - \delta_2 M_i^2 \leq b_i^2, i = 1, \ldots, m_2$$
$$\delta_1 + \delta_2 = 1$$
$$\delta_1, \delta_2 \in \{0, 1\}$$

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provided that the parameters M_i^j satisfy
$f_i^j(x) \leq b_i^j + M_i^j, i = 1, \ldots, m_j, j = 1, 2$

This can be used to define nonconvex polygonal feasible sets.

Conditional constraints 1

A conditional constraint of the form

$$f(x) > a \Longrightarrow g(x) \leq b$$

can be modeled with the alternative set of constraints

$$f(x) \leq a \quad \text{and/or} \quad g(x) \leq b$$

which in turn can be modeled as explained before (see more equivalences for conditional statements later on).

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K out of N constraints must hold

If we have a set of N constraints

$$f_1(x) \leq b_1, \ldots, f_N(x) \leq b_N$$

and only K out of the N constraints must hold, this can be modeled as follows:

$$f_1(x) \leq b_1 + M_1\delta_1$$
$$\ldots$$
$$f_N(x) \leq b_N + M_N\delta_N$$
$$\sum_{i=1}^{N} \delta_i = N - K$$
$$\delta_i \in \{0,1\}, i = 1, \ldots, N$$

where M_i is an upper bound for $f_i(x) - b_i$.

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Modeling fixed costs

The discontinuous function to be minimized

$$\min f(x) = \begin{cases} 0 & \text{if } x = 0 \\ k + g(x) & \text{if } 0 < x \leq b \end{cases}$$

which sets a fixed cost k in case the variable x is used (in case $x > 0$) can be written as

$$\begin{array}{ll} \min & k\delta + g(x) \\ s.t. & x \leq b\delta \\ & x \geq 0 \\ & \delta \in \{0, 1\} \end{array}$$

Notice that

$$\delta = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

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Let x be a continuous variable such that $L \leq x \leq U$. And let $\delta \in \{0, 1\}$ be a binary variable.

MILP: modeling tricks

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Conditional constraints 2

$$\delta = 0 \Longrightarrow x \leq 0$$

can be modeled as

$$x \leq \delta U.$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$x > 0 \Longrightarrow \delta = 1$$

MILP: modeling tricks

Let x be a continuous variable such that $L \leq x \leq U$. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 3

$$\delta = 0 \Longrightarrow x \geq 0$$

can be modeled as

$$x \geq \delta L.$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$x < 0 \Longrightarrow \delta = 1$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 4 (type $\leq$)

$$\delta = 1 \Longrightarrow f(x) \leq b$$

can be modeled as

$$f(x) \leq b + M(1 - \delta).$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$f(x) > b \Longrightarrow \delta = 0$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 5 (type $\leq$)

$$f(x) \leq b \Longrightarrow \delta = 1$$

is equivalent to

$$\delta = 0 \Longrightarrow f(x) > b$$

which can be tranformed into

$$\delta = 0 \Longrightarrow f(x) \geq b + \epsilon.$$

The previous expressions can be both modeled as

$$f(x) \geq b + \epsilon + (m - \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

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Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 6 (type $\geq$)

$$\delta = 1 \Longrightarrow f(x) \geq b$$

can be modeled as

$$f(x) \geq b + m(1 - \delta).$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$f(x) < b \Longrightarrow \delta = 0$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 7 (type $\geq$)

$$f(x) \geq b \Longrightarrow \delta = 1$$

is equivalent to

$$\delta = 0 \Longrightarrow f(x) < b$$

which can be transformed into

$$\delta = 0 \Longrightarrow f(x) \leq b - \epsilon.$$

The previous expressions can be both modeled as

$$f(x) \leq b - \epsilon + (M + \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 8 (type $=$)

$$\delta = 1 \Longrightarrow f(x) = b \text{ is equivalent to } \delta = 1 \Longrightarrow \begin{cases} f(x) \leq b \\ f(x) \geq b \end{cases}$$

Hence, it can be modeled by the constraints

$$\begin{aligned} f(x) &\leq b + M(1 - \delta) \\ f(x) &\geq b + m(1 - \delta) \end{aligned}$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$f(x) \neq b \Longrightarrow \delta = 0$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 9 (type $=$)

$$f(x) = b \implies \delta = 1 \quad \text{is equivalent to} \quad \begin{array}{c} f(x) \leq b \implies \delta_1 = 1 \\ f(x) \geq b \implies \delta_2 = 1 \\ \left.\begin{array}{l} \delta_1 = 1 \\ \delta_2 = 1 \end{array}\right\} \implies \delta = 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{array}$$

which can be modeled as

$$\begin{array}{c} f(x) \geq b + \epsilon + (m - \epsilon)\delta_1 \\ f(x) \leq b - \epsilon + (M + \epsilon)\delta_2 \\ \delta_1 + \delta_2 - \delta \leq 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{array}$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 9 (type $=$)

Since $f(x) = b \Longrightarrow \delta = 1$ is equivalent to $\delta = 0 \Longrightarrow f(x) \neq b$

this last conditional constraint can also be modeled as

$$\begin{gathered} f(x) \geq b + \epsilon + (m - \epsilon)\delta_1 \\ f(x) \leq b - \epsilon + (M + \epsilon)\delta_2 \\ \delta_1 + \delta_2 - \delta \leq 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{gathered}$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 10: double implications

Double implications can be transformed into two unidirectional implications. For instance

$$\delta = 1 \iff f(x) \leq b$$

is equivalent to

$$\begin{cases} \delta = 1 & \implies & f(x) \leq b \\ f(x) \leq b & \implies & \delta = 1 \end{cases}$$

Hence, it can be modeled as

$$f(x) \leq b + M(1 - \delta)$$
$$f(x) \geq b + \epsilon + (m - \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 10: double implications

$$\delta = 1 \Longleftrightarrow f(x) \geq b$$

can be modeled as

$$f(x) \geq b + m(1 - \delta)$$
$$f(x) \leq b - \epsilon + (M + \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 10: double implications

$$\delta = 1 \iff f(x) = b$$

can be modeled as

$$\begin{gathered} f(x) \leq b + M(1-\delta) \\ f(x) \geq b + m(1-\delta) \\ f(x) \geq b + \epsilon + (m - \epsilon)\delta_1 \\ f(x) \leq b - \epsilon + (M + \epsilon)\delta_2 \\ \delta_1 + \delta_2 - \delta \leq 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{gathered}$$

MILP: modeling tricks

Equivalences for conditional propositions

The following equivalences can be used before converting them into constraints:

$P \Rightarrow Q$	$\neg P \vee Q$
$P \Rightarrow (Q \wedge R)$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
$P \Rightarrow (Q \vee R)$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$
$(P \wedge Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$

MILP: modeling tricks

Assume that the indicator variable δ_i is equal to 1 when the constraint C_i holds:

$$\delta_i = \begin{cases} 1 & \text{if } C_i \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

Simple conditional or composed statements

$C_1 \vee C_2$	$\delta_1 + \delta_2 \geq 1$
$C_1 \wedge C_2$	$\delta_1 + \delta_2 = 2$
$\neg C_1$	$\delta_1 = 0$
$C_1 \implies C_2$	$\delta_1 \leq \delta_2$
$C_1 \iff C_2$	$\delta_1 = \delta_2$

MILP: modeling tricks

Complex conditional or composed statements

Complex conditional or composed statements are decomposed into two implications in order to model them easier.

Example

$$(C_1 \vee C_2) \Longrightarrow (C_3 \vee C_4 \vee C_5)$$

is modeled as

$$(\delta_1 + \delta_2 \geq 1) \Longrightarrow (\delta_3 + \delta_4 + \delta_5 \geq 1)$$

which, in turn, can be transformed into

$$(\delta_1 + \delta_2 \geq 1) \Rightarrow \delta = 1 \Rightarrow (\delta_3 + \delta_4 + \delta_5 \geq 1)$$

or more clearly,

$$\begin{cases} (\delta_1 + \delta_2 \geq 1) \Rightarrow \delta = 1 \\ \delta = 1 \Rightarrow (\delta_3 + \delta_4 + \delta_5 \geq 1) \end{cases}$$

which becomes

$$\begin{cases} \delta_1 + \delta_2 \leq 2\delta \\ \delta \leq \delta_3 + \delta_4 + \delta_5 \end{cases}$$

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Example

$$(x \leq b) \wedge (x \geq 1) \Longrightarrow (y = z + 1)$$

is first transformed into

$$(x \leq b) \wedge (x \geq 1) \Longrightarrow \delta = 1 \Longrightarrow (y = z + 1)$$

and this in turn is written as

$$\begin{array}{lll}
(x \leq b) \Rightarrow \delta_1 = 1 & & x \geq b + \epsilon + (m_1 - \epsilon)\delta_1 \\
(x \geq 1) \Rightarrow \delta_2 = 1 & & x \leq 1 - \epsilon + (M_1 + \epsilon)\delta_2 \\
(\delta_1 = 1) \wedge (\delta_1 = 1) \Rightarrow \delta = 1 & \text{which becomes} & \delta_1 + \delta_2 - \delta \leq 1 \\
(\delta = 1) \Rightarrow (y \geq z + 1) & & y - z \geq 1 + m_2(1 - \delta) \\
(\delta = 1) \Rightarrow (y \leq z + 1) & & y - z \leq 1 + M_2(1 - \delta)
\end{array}$$

where $\epsilon > 0$ is a small number and $m_1 \leq x - b$, $M_1 \geq x - 1$, $m_2 \leq y - z - 1 \leq M_2$.

MILP: modeling tricks

More tricks have been designed to:

- Define nonconvex polygonal regions throught a set of constraints.
- Work with Special Ordered Sets of type 1 (SOS1), where in a set of variables only one of them can have a value different from 0, and SOS2, where in a set a variables at most two of them can be different from 0 and they must be consecutive variables.

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Sometimes the same problem can be modeled in different ways. A formulation 'A' is said to be better (stronger) than another formulation 'B' if the feasible set of 'A' is included in the feasible set of 'B'.

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Interestingly, in MILP sometimes it is better a formulation with a bigger number of variables and constraints!