Variational Methods in Image Segmentation

Zoltan Kato

Image Processing & Computer Graphics Dept.
University of Szeged
Hungary

Road Map

- Mumford-Shah energy functional
  - Segmentation as optimal approximation
  - Cartoon model
- Level Set representation
  - Implicit contour representation
  - Motion under curvature
- "Chan and Vese" model
  - Relation to the Cartoon model
  - Constructing a Level Set representation
  - Demo

Mumford-Shah Functional

- Proposed in their influential paper by
  - David Mumford
    - http://www.dam.brown.edu/people/mumford/
  - Jayant Shah
    - http://www.math.neu.edu/~shah/

Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems. *Communications on Pure and Applied Mathematics, Vol. XLII, pp 577-685, 1989*

Images as functions

- A gray-level image represents the light intensity recorded in a plan domain ***R***
  - We may introduce coordinates ***x***, ***y***
  - Let ***g(x,y)*** denote the intensity recorded at the point ***(x,y)*** of ***R***
  - The function ***g(x,y)*** defined on the domain ***R*** is called an image.

What kind of function is *g*?

- The light reflected by the surfaces S_i of various objects O_i will reach the domain R in various open subsets R_i
- When O_1 appears as the background to the sides of O_2 then the open sets R_1 and R_2 will have a common boundary (**edge**)
- One usually expects $g(x,y)$ to be discontinuous along this boundary

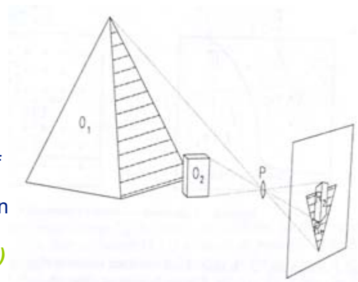

Figure 1. An *image* of a 3D scene.

Figure from D. Mumford & J. Shah: **Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems**. *Communications on Pure and Applied Mathematics, Vol. XLII, pp 577-685, 1989*

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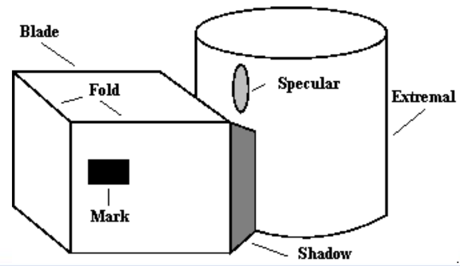
Other discontinuities

- Surface orientation of visible objects (cube)
- Surface markings
- Illumination (shadows, uneven light)



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Piece-wise smooth *g*

- In all cases, we expect $g(x,y)$ to be piece-wise smooth to the first approximation.
- It is well modelled by a set of smooth functions f_i defined on a set of disjoint regions R_i covering R.
- Problems:
  - Textured objects (regions perceived homogeneous but lots of discontinuities in intensity)
  - Sahdows are not true discontinuities
  - Partially transparent objects
  - Noise
- Still widely and succesfully applied model!

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Segmentation problem

- Consists in computing a decomposition of the domain of the image $g(x,y)$

$$R = \bigcup_{i=1}^{n} R_i$$

1. g varies smootly and/or slowly within R_i
2. g varies discontinuously and/or rapidly across most of the boundary Γ between regions R_i

Optimal approximation

- Segmentation problem may be restated as
  - finding optimal approximations of a general function ***g***
  - by a piece-wise smooth function ***f***, whose restrictions f_i to the regions R_i are **differentiable**
- Many other applications:
  - Speech recognition
  - Sonar, radar or laser range data
  - CT scans
  - etc…

Optimal segmentation

- Mumford and Shah studied 3 functionals which measure the degree of match between an image ***g(x,y)*** and a segmentation.
- First, they defined a general functional ***E*** (the famous Mumford-Shah functional):
  - R_i will be disjoint connected open subsets of the planar domain ***R***, each one with a piece-wise smooth boundary
  - Γ will be the union of the boundaries.

$$R = \coprod_{i=1}^{n} R_i \coprod \Gamma$$

Mumford-Shah functional

- Let ***f*** differentiable on $\cup R_i$ and allowed to be discontinuous across Γ.

$$E(f,\Gamma) = \mu^2 \iint_R (f-g)^2\,dxdy + \iint_{R-\Gamma} \|\nabla f\|^2\,dxdy + \nu|\Gamma|$$

- The smaller ***E***, the better ***(f, Γ)*** segments ***g***

1. ***f*** approximates ***g***
2. ***f*** (hence ***g***) does not vary much on R_is
3. The boundary Γ be as short as possible.

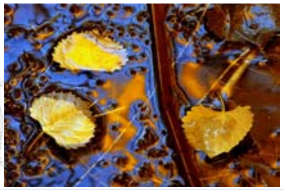
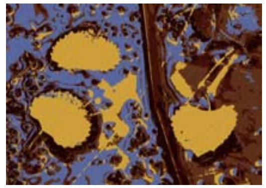
- Dropping any term would cause ***inf E=0***.

Cartoon image

- ***(f, Γ)*** is simply a cartoon of the original image ***g***.
  - Basically ***f*** is a new image with edges drawn sharply.
  - The objects are drawn smootly without texture
  - ***(f, Γ)*** is essentially an idealization of ***g*** by the sort of image created by an artist.
  - Such cartoons are perceived correctly as representing the same scane as ***g*** ➔ ***f*** is a simplification of the scene containing most of its essential features.

Cartoon image example



Piecewise constant approximation

- A special case of E where $f=a_i$ is constant on each open set R_i.

$$\mu^{-2}E(f,\Gamma)=\sum_i \iint_{R_i}(g-a_i)^2\,dxdy+\frac{\nu}{\mu^2}|\Gamma|$$

- Obviously, it is minimized in a_i by setting a_i to the mean of g in R_i:

$$a_i=mean_{R_i}(g)=\iint_{R_i} gdxdy \Big/ area(R_i)$$

Piecewise constant approximation

$$E_0(\Gamma)=\sum_i \iint_{R_i}(g-mean_{R_i}(g))^2\,dxdy+\frac{\nu}{\mu^2}|\Gamma|$$

- It can be proven that minimizing E_0 is well posed:
  - For any continuous g, there exists a Γ made up of finit number of singular points joined by a finit number of arcs on which E_0 atteins a minimum.
- It can also be shown that E_0 is the natural limit functional of E as $\mu \to 0$

Road Map

- Mumford-Shah energy functional
  - Segmentation as optimal approximation
  - Cartoon model
- Level Set representation
  - Implicit contour representation
  - Motion under curvature
- "Chan and Vese" model
  - Relation to the Cartoon model
  - Constructing a Level Set representation
  - Demo

Level Set Representation

- Developed by
  - Stanley Osher
    - http://www.math.ucla.edu/~sjo/
  - J. A. Sethian
    - http://math.berkeley.edu/~sethian/
- J. A. Sethian: Level Set Methods and Fast Marching Methods. *Cambridge University Press, 1999*.

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What is it?

- It is a generic numerical method for evolving fronts in an implicit form.
  - It handles topological changes of the evolving interface
  - Define problem in 1 higher dimension
    - Seems crazy but it well worth the extra effort…
- Use an implicit representation of the contour ***C*** as the zero level set of higher dimensional function ϕ ***- the level set function***

$$\phi(C) = 0$$

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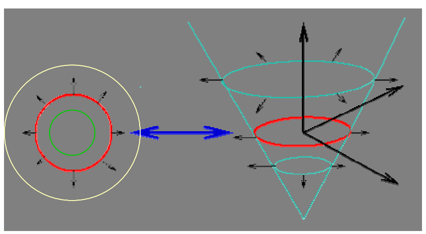
How it works?

- Move the level set function, $\phi(x,y,t)$, so that it rises, falls, expands, etc.
- Contour = cross section at ***z = 0***



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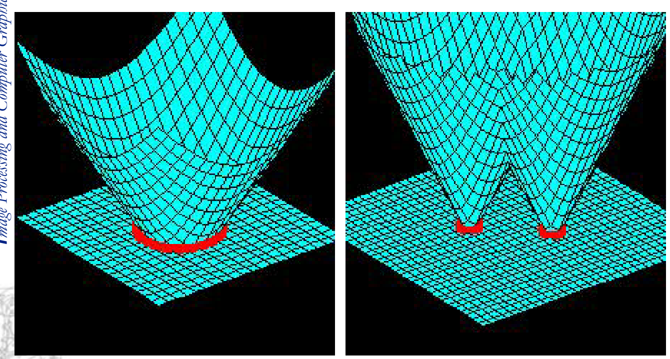
Implicit curve evolution



Images taken from the level set website: http://math.berkeley.edu/~sethian/Explanations/level_set_explain.html/

How to Move the Level Set Surface?

- Define a velocity field ***F***, that specifies how contour points move in time
  - Based on application-specific physics such as time, position, normal, curvature, image gradient magnitude
- Build an initial value for the level set function, ***ϕ(x,y,t=0)***, based on the initial contour position
- Adjust ϕ over time; current contour defined by ***ϕ(x(t), y(t), t) = 0***

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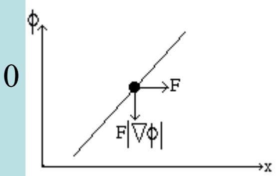
The Level Set Evolution Equation

- Manipulate ϕ to indirectly move ***C***:

$$\phi(C) = 0$$

$$\frac{d\phi(C)}{dt} = \frac{\partial C}{\partial t} \cdot \nabla\phi + \frac{\partial \phi}{\partial t} = 0$$

$$\frac{\partial \phi}{\partial t} = -F|\nabla\phi|$$



where ***F*** is the speed function normal to the curve

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Example: an expanding circle

- Level Set representation of a circle:

$$\phi(x,y) = \sqrt{x^2+y^2} - r$$

  - Setting ***F = 1*** causes the circle to expand uniformly
  - Observe that ***∇ϕ = 1*** almost everywhere (by choice of representation), so we obtain the level set evolution equation:

$$\frac{\partial \phi}{\partial t} = -1$$

- Explicit solution:

$$\phi(x,y,t) = \sqrt{x^2+y^2} - r - t$$

which means that the circle has radius ***r + t*** at time ***t***, as expected!

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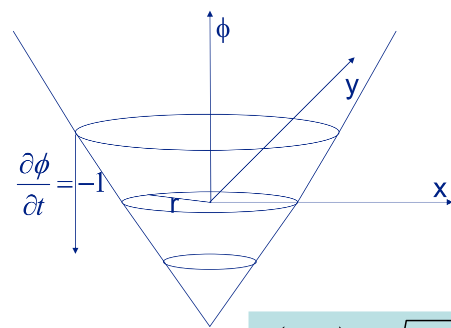
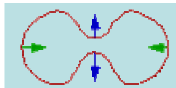
Example: an expanding circle



$$\phi(x,y) = \sqrt{x^2+y^2} - r$$

Motion under curvature

- What about more complicated shapes?



- ***Motion under curvature***: each piece of the curve moves perpendicular to the curve with speed proportional to the curvature.
  - Since the curvature can be either positive or negative (depending on whether the curve is turning clockwise or counterclockwise), some parts of the curve move outwards while others move inwards.

Motion under curvature

- A famous theorem in differential geometry (proved in the 90's), says that:
  - any simple closed curve moving under its curvature collapses to a circle and then disappears.

Images taken from the level set website: http://math.berkeley.edu/~sethian/Explanations/level_set_explain.html/

Level Set Segmentation

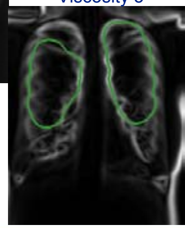
- Since the choice of ϕ is somewhat arbitrary, we can choose a signed distance function from the contour.
  - This distance function is negative inside the curve and positive outside.
  - A distance function is chosen because it has unit gradient almost everywhere and so is smooth.
- By choosing a suitable speed function ***F***, we may segment an object in an image

Level Set Segmentation

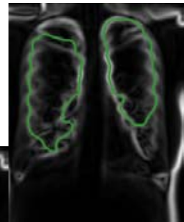
- The standard level set segmentation speed function is: $$F = 1 - \varepsilon\kappa + \beta\left(\nabla\phi \cdot \nabla\left|\nabla I\right|\right)$$
- The ***1*** causes the contour to inflate inside the object
- The $-\varepsilon\kappa$ (viscosity) term reduces the curvature of the contour
- The final term (edge attraction) pulls the contour to the edges
- Imagine this speed function as a balloon inflating inside the object. The balloon is held back by its edges, and where there are holes in the boundary it bulges but is halted by the viscosity ε.

Segmentation Example

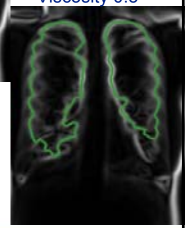
Lung x-ray



Viscosity 5



Viscosity 2



Viscosity 0.5

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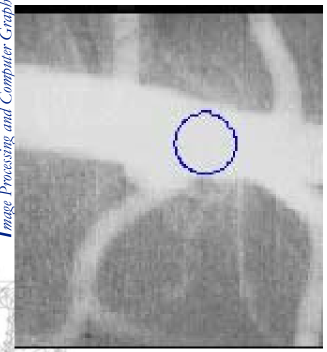
Some more examples



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”Chan and Vese” model

- Tony F. Chan & Luminita Vese: **Active Contours without Edges**. *IEEE Transactions on Image Processing, 10(2), pp 266-277, Feb. 2001*.
- Construct a model to segment an image into *foreground* and *background* regions (binarization) based on intensities instead of image gradient.

Energy functional

$$F(c_1,c_2,C)=\mu L(C)+\nu A(in(C))$$
$$+\lambda_1\int_{in(C)}|u_0(x,y)-c_1|^2+\lambda_2\int_{out(C)}|u_0(x,y)-c_2|^2$$

- c_1 and c_2 are the average intensity levels inside and outside of the contour
- The minimization problem:

$$\inf_{c_1,c_2,C} F(c_1,c_2,C)$$

Relation with the Mumford-Shah functional

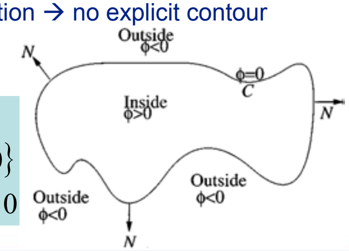
- The "Chan and Vese" model is a special case of the Mumford Shah model (minimal partition problem)
  - it looks for the best approximation of u_0, as a function u taking only two values c_1 and c_2
  - $\nu=0$ and $\lambda_1=\lambda_2=\lambda$
  - u=average(u_0 in/out)
  - C is the CV active contour
- "Cartoon" model (piece-wise constant approximation):

$$F^{MS}(u,C)=\mu L(C)+\lambda\int_\Omega|u_0-u|^2$$

Level set formulation

- Considering the disadvantages of explicit contour representation, the model is solved using level set formulation
  - level set representation → no explicit contour

$$C=\{(x,y)\in\Omega:\phi(x,y)=0\}$$
$$in(C)=\{(x,y)\in\Omega:\phi(x,y)>0\}$$
$$out(C)=\{(x,y)\in\Omega:\phi(x,y)<0\}$$



Replacing C with Φ

- Using the Heaviside (sign) and Dirac measure (PSF) functions:

$$H(z)=\begin{cases}1, & \text{if } z\geq 0\\ 0, & \text{if } z<0,\end{cases} \qquad \delta_0(z)=\frac{d}{dz}H(z)$$

- We get

$$L\{\phi=0\}=\int_\Omega|\nabla H(\phi(x,y))|=\int_\Omega\delta(\phi(x,y))|\nabla\phi(x,y)|$$
$$A\{\phi\geq 0\}=\int_\Omega H(\phi(x,y))$$

Replacing C with Φ

- The intensity terms

$$\int_{in(C)} |u_0(x,y)-c_1|^2 dxdy = \int_{\phi>0} |u_0(x,y)-c_1|^2 dxdy = \int_{\Omega} |u_0(x,y)-c_1|^2 H(\phi(x,y)) dxdy$$

$$\int_{out(C)} |u_0(x,y)-c_2|^2 dxdy = \int_{\phi<0} |u_0(x,y)-c_2|^2 dxdy = \int_{\Omega} |u_0(x,y)-c_2|^2 (1-H(\phi(x,y))) dxdy$$

Average intensities

- We can express c_1 and c_2 as functions of Φ

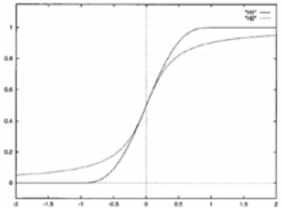
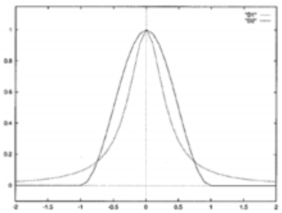
$$c_1(\phi) = \frac{\int_{\Omega} u_0(x,y) H(\phi(x,y)) dxdy}{\int_{\Omega} H(\phi(x,y)) dxdy}$$

$$c_2(\phi) = \frac{\int_{\Omega} u_0(x,y)(1-H(\phi(x,y))) dxdy}{\int_{\Omega} (1-H(\phi(x,y))) dxdy}$$

- Note that $H(\Phi)$ is the ***characteristic function*** of the foreground regions!

Level set formulation of the model

- Combining the presented energy terms we can write the Chan and Vese functional as a function of Φ.
- Minimization F wrt. Φ → gradient descent
- In order to compute the associated Euler–Lagrange equation, we consider slightly regularized versions of the functions H and δ.

Euler-Lagrange equation

- Regularization used:

$$H_{2,\varepsilon}(z) = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left(\frac{z}{\varepsilon}\right)\right). \qquad \delta_\varepsilon = H'_\varepsilon$$

- Euler-Lagrange equation:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)\left[\mu\kappa(\phi)|\nabla\phi| - \nu - \lambda_1(u_0-c_1)^2 + \lambda_2(u_0-c_2)^2\right]$$

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The algorithm

- **Initialization n=0**
- **repeat**
  - **n++**
  - **Computing c_1 and c_2**
  - **Evolving the level-set function**
- **until the solution is stationary, or $n>n_{max}$**

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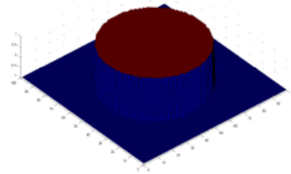
Initialization

- We set the values of the level set function
  - outside = -1
  - inside = 1
- Any shape can be the initialization shape

```
init()
  for all (x, y) in Phi
    if (x, y) is inside
      Phi(x, y)=1;
    else
      Phi(x, y)=-1;
    fi;
  end for
```



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Computing c_1 and c_2

- The mean intensity of the image pixels inside and outside

```
colors()
  out = find(Phi < 0);
  in = find(Phi > 0);
  c1 = sum(Img(in)) / size(in);
  c2 = sum(Img(out)) / size(out);
```


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Approximation of the Curvature

$$\kappa(\phi) = \frac{\phi_{xx}\phi_y^2 - 2\phi_{xy}\phi_x\phi_y + \phi_{yy}\phi_x^2}{(\phi_x^2+\phi_y^2)^{3/2}}$$

$$\phi_x = \frac{\phi_{p0} - \phi_{m0}}{2\Delta s} \qquad \phi_y = \frac{\phi_{0p} - \phi_{0m}}{2\Delta s}$$

$$\phi_{xx} = \frac{\phi_{p0} + \phi_{m0} - 2\phi_{00}}{\Delta s^2}$$

$$\phi_{yy} = \frac{\phi_{0p} + \phi_{0m} - 2\phi_{00}}{\Delta s^2}$$

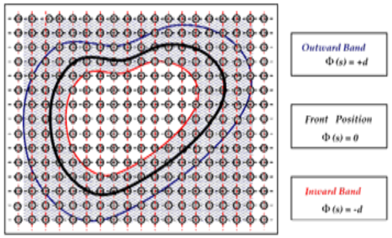
$$\phi_{xy} = \frac{\phi_{pp} - \phi_{mp} - \phi_{pm} + \phi_{mm}}{4\Delta s^2}$$

Finite differences

```
for all (x, y)
  fx(x, y) = (Phi(x+1, y)-Phi(x-1, y))/(2*delta_s);
  fy(x, y) =…
  fxx(x, y) =…
  fyy(x, y) =…
  fxy(x, y) =…
```

delta_s recommended between 0.1 and 1.0

Narrow band



It is useful to compute the level set function not on the whole image domain but in a narrow band near to the contour.
Abs(ϕ)<d

Decreasing the computational complexity.

Narrow band

- **Initialization n=0**
- **repeat**
  - **n++**
  - **Determination of the narrow band**
  - **Computing c_1 and c_2**
  - **Evolving the level-set function on the narrow band**
  - **Re-initialization**
- **until the solution is stationary, or $n>n_{max}$**

Re-initialization

- Optional step

$$a=\frac{\phi(x,y)-\phi(x-1,y)}{h};b=\frac{\phi(x+1,y)-\phi(x,y)}{h};$$

$$c=\frac{\phi(x,y)-\phi(x,y-1)}{h};d=\frac{\phi(x,y+1)-\phi(x,y)}{h};$$

$$a^{+}=\max(a,0);a^{-}=\min(a,0);\ldots$$

$$G=\begin{cases}\sqrt{\max(a^{+2},b^{-2})+\max(c^{+2},d^{-2})}-1 & ,\phi(x,y)>0\\ \sqrt{\max(a^{-2},b^{+2})+\max(c^{-2},d^{+2})}-1 & ,\phi(x,y)<0\end{cases}$$

$$\phi=\phi-\Delta t\cdot sign(\phi)\cdot G;$$

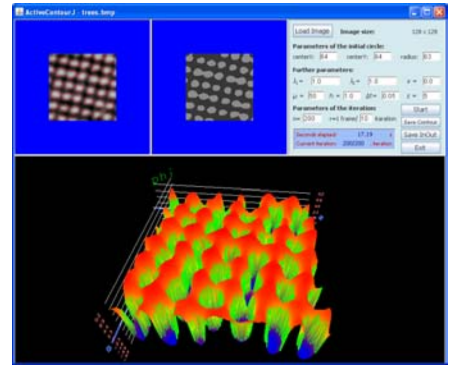
h is a normalizing term recommended between 0.1 and 2.
Δt is the time step, see above!

Stop criteria

- Stop the iterations if:
  - The maximum iteration number were reached
  - Stationary solution:
    - The energy is not changing
    - The contour is not moving
    - …

Demonstration program

- Parameters:
  - ***Δt*** is recommended between 0.01 and 0.9. Be careful $\Delta t<1$!
  - ***h*** is a normalizing term recommended between 0.1 and 2.
  - ***ε*** is the regularizing parameter



ActiveContourJ software courtesy **Laszlo Csernetics**