Utilizing the discrete orthogonality of Zernike functions in corneal measurements

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Overview of the presentation

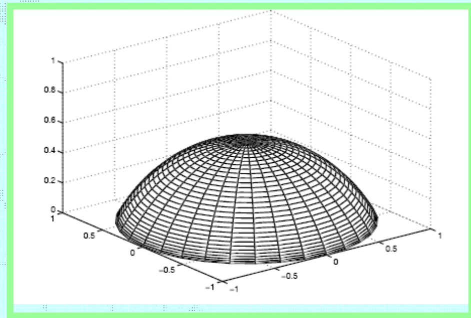
- Introduction
  - reflective corneal topography and its measurement patterns
  - orthogonal system of Zernike functions
  - the use Zernike functions in ophthalmology
  - discrete orthogonal system of Zernike functions
- Continuous Zernike functions
  - their relation with Jacobi polynomials
- Discretization of Zernike functions
  - mesh ensuring the discrete orthogonality of Zernike functions
  - significance of the quadrature formulas in discretization
- The discrete Zernike coefficients
  - program-implementation
  - precision achieved
  - examples
- Conclusions and future work

Introduction – Reflective corneal topography and its measurement patterns 1/7

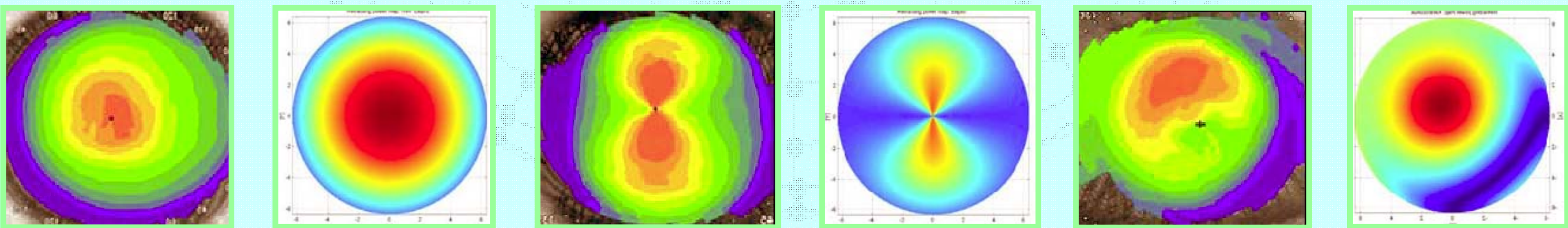
- Eye, cornea
  - in technical terms, the ***human eye*** can be considered an *imaging sensor* with its *frontal section* being responsible for focusing the incoming light rays, while
  - its *rear section* is responsible for converting the image formed on its internal surface into electrical signals for further processing

  - the ***cornea*** – located in its frontal section – is the primary optical structure of the human eye
  - the corneal tissue is *transparent*
  - the human cornea is an optical structure, which generates about the *70% of the total refractive power* of the eye
  - ***other structures*** in the *light-path* – including the crystalline lens – contribute less total refractive power

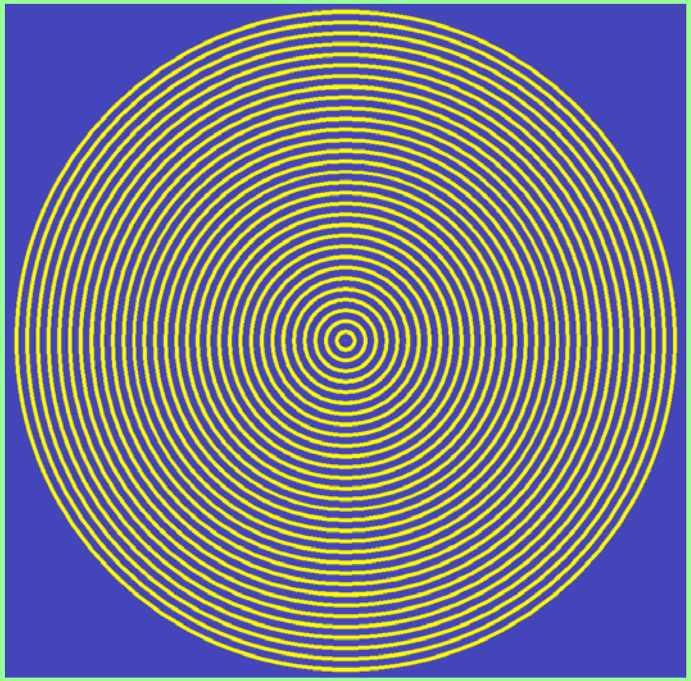
Introduction – Reflective corneal topography and its measurement patterns 2/7

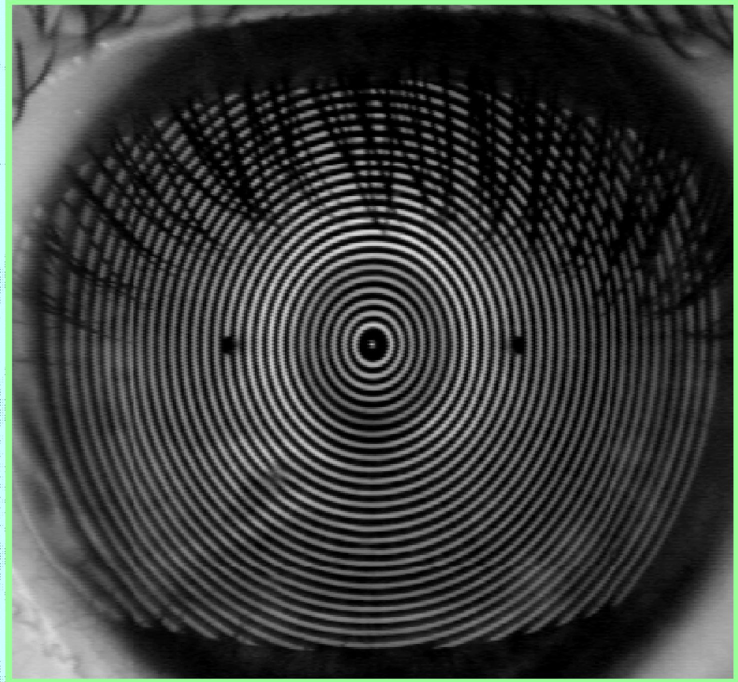


- The corneal surface
  - often ***modelled*** as a *spherical calotte*
  - there are *more complex models,* as well
- Purpose of a cornea topographic examination
  - ***determine*** and display the *shape* and the *refractive power* of the living cornea
  - due to the *high refractive power* of the cornea, the *detailed topography* is of great diagnostic importance

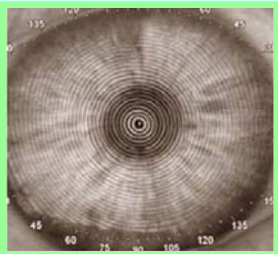


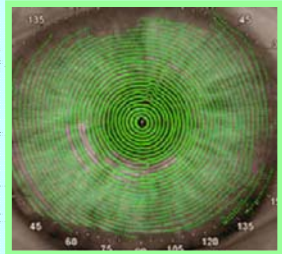
Introduction – Reflective corneal topography and its measurement patterns 3/7

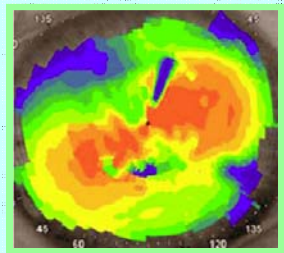




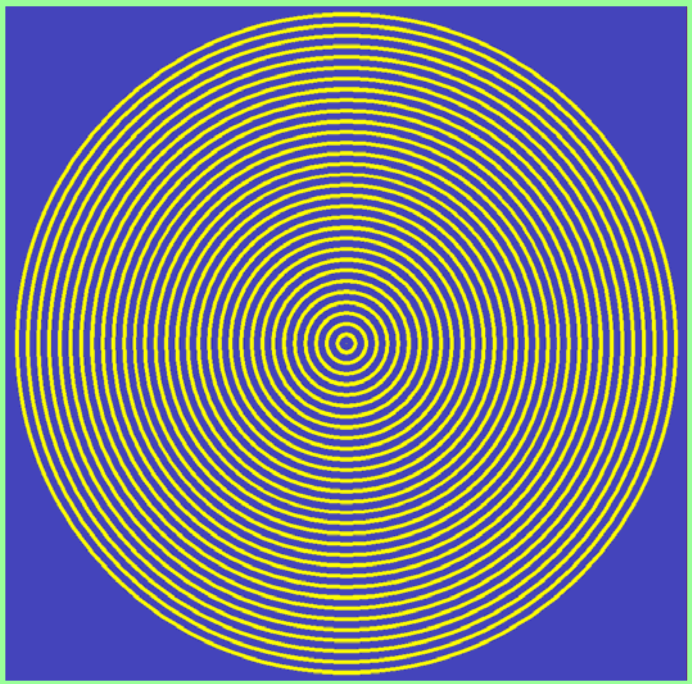
The *measurement properties* of the ***conventional Placido disk-based topographers*** are rather problematic, as *no point correspondences* are available for the purpose of the geometrical surface reconstruction.

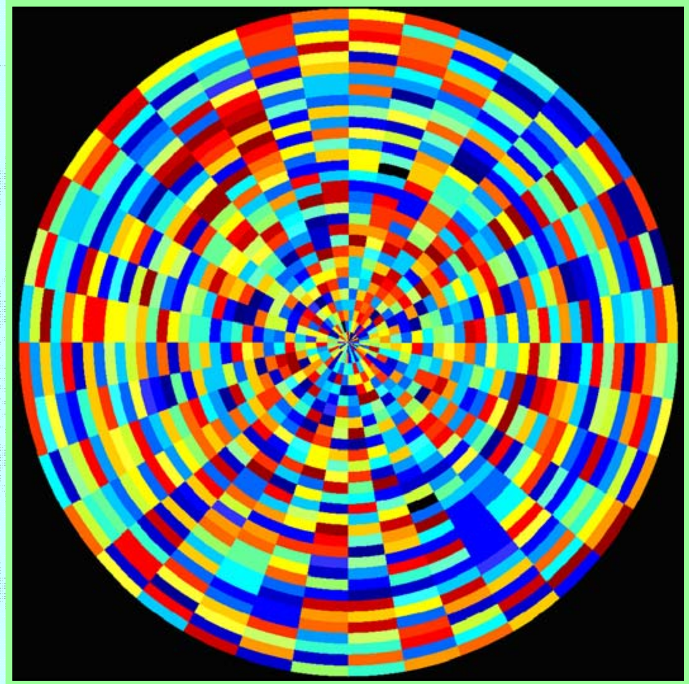






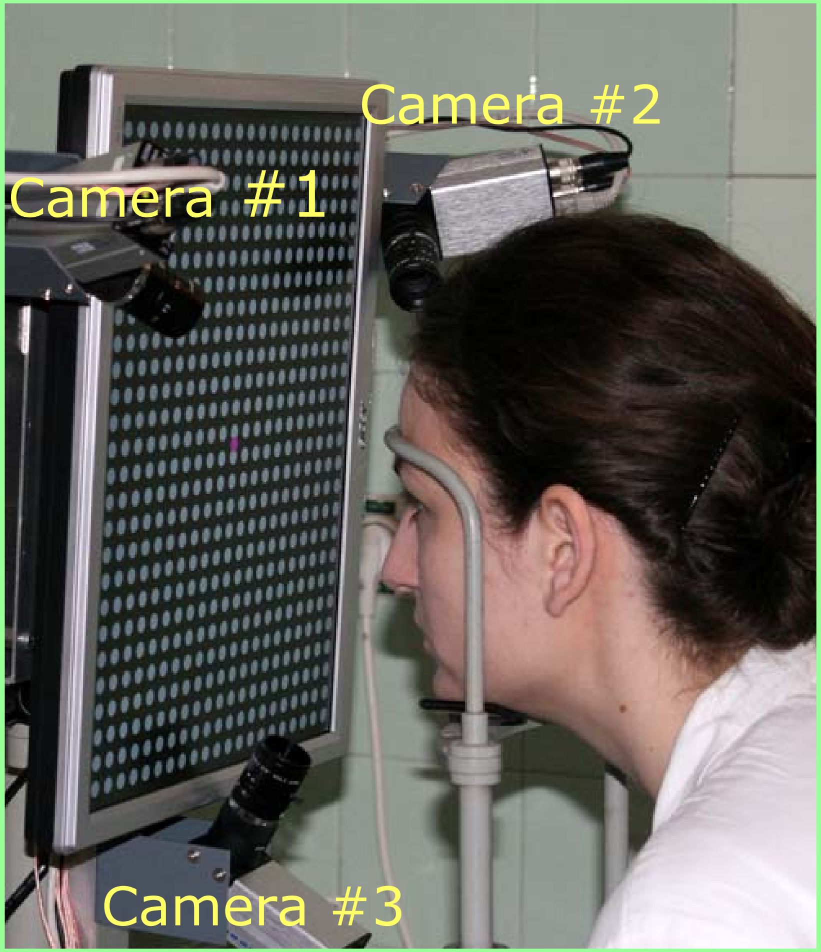
Introduction – Reflective corneal topography and its measurement patterns

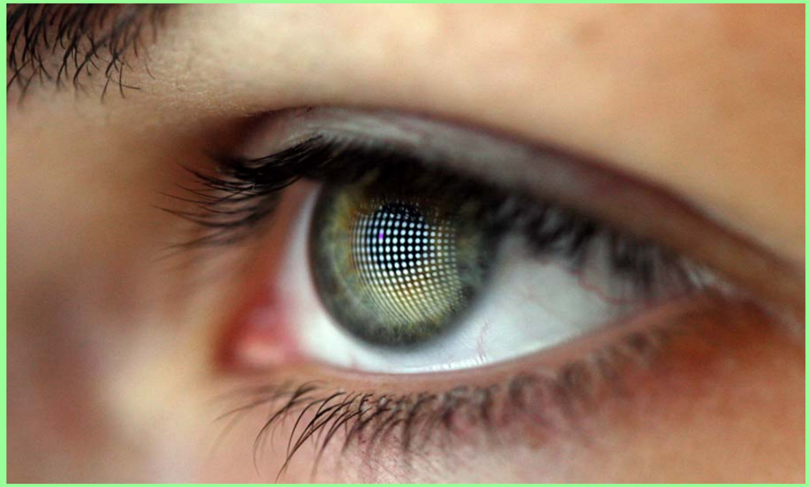




Using some *sophisticated version* of Placido disk, e.g., a ***random-coloured Placido disk,*** *point correspondences* can be found for reconstruction.

Introduction – Reflective corneal topography and its measurement patterns 5/7

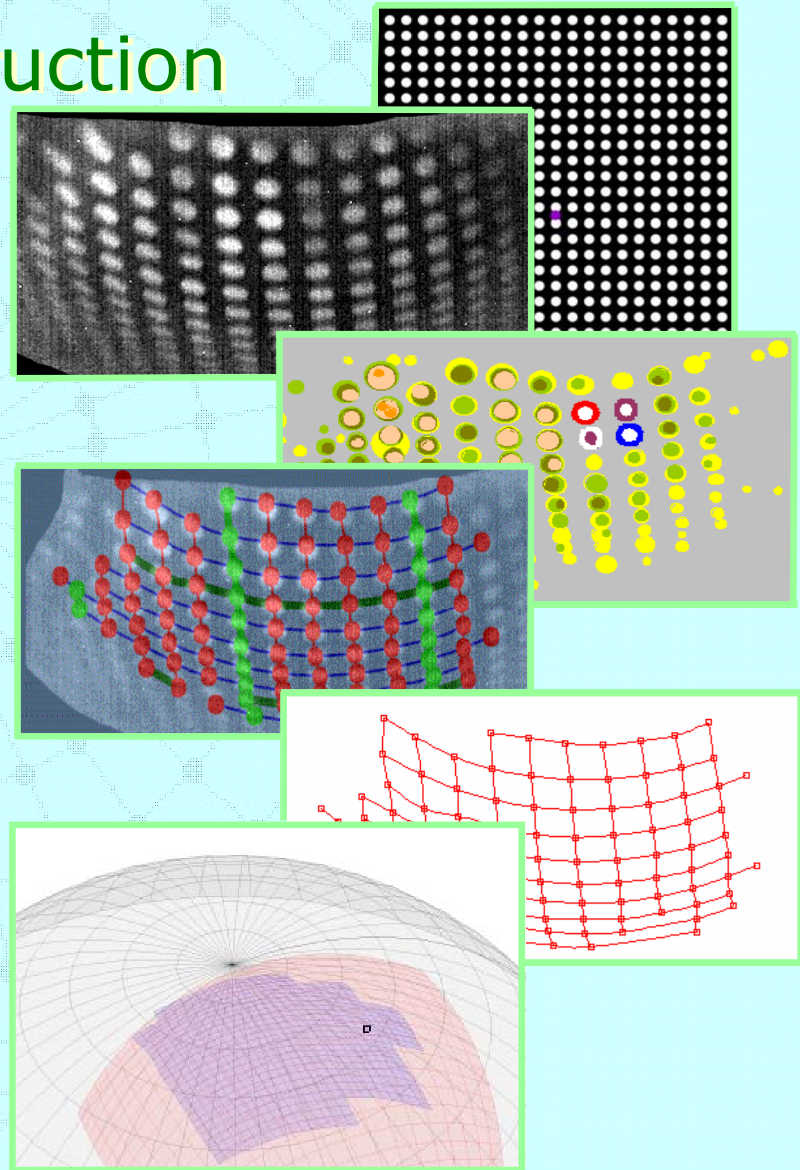




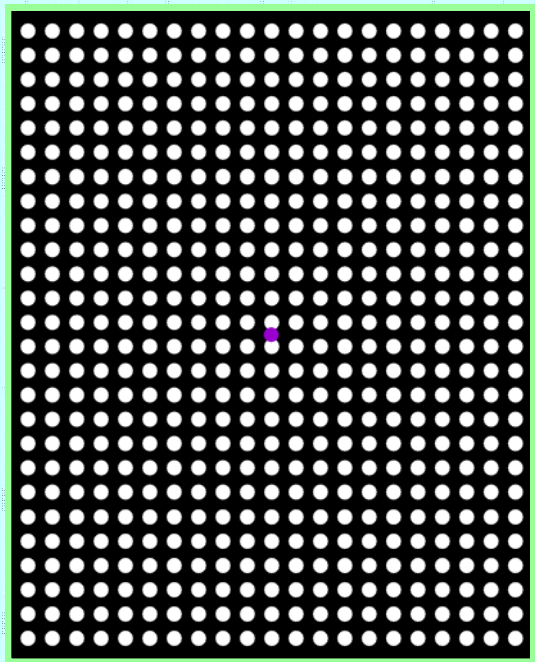
Recently, a ***multi-camera surface reconstruction method*** was proposed for the purpose of corneal topography. The reconstruction is achieved by solving the *partial differential equations* (PDE's) describing the specular reflections at the corneal surface.

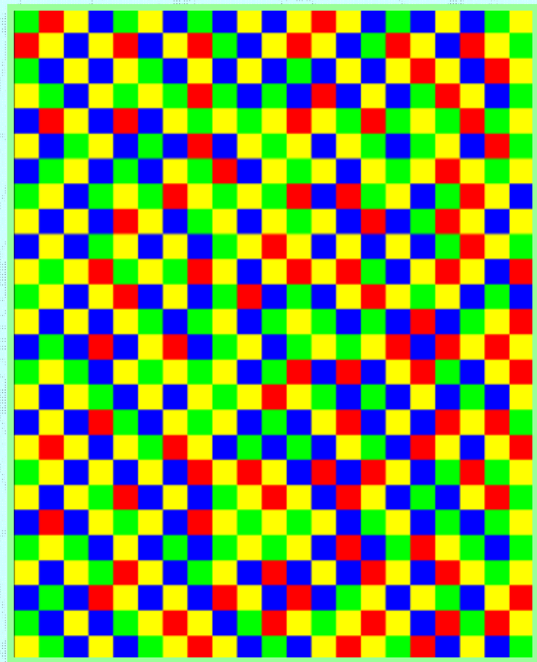
Introduction – Reflective corneal topography and its measurement patterns

- The process of reconstruction
  - *image* segmentation
  - *blob* filtering
  - blob identification
  - *spline* approximation
  - numerical integration of the first-order *ordinary differential equations*
  - derived from the first-order *PDE's* via appropriate *parametrization*



Introduction – Reflective corneal topography and its measurement patterns

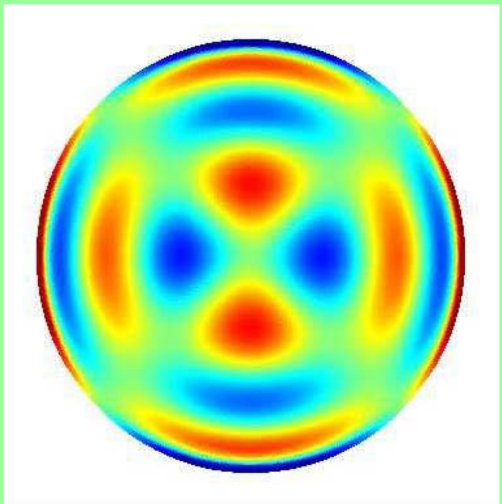


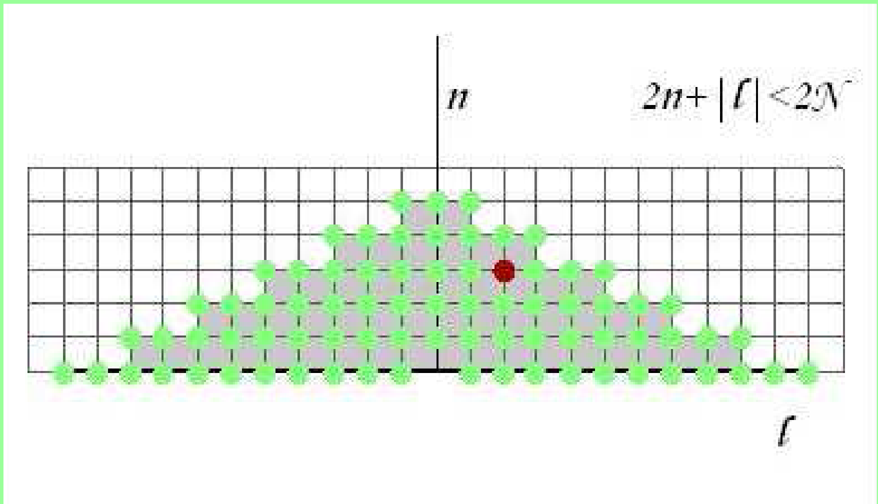


The ***measurement patterns*** used in the multi-camera topographer arrangement: *a square grid of circular spots* with its centre marked and a *position-coding colour checkerboard.*

Introduction – The orthogonal system of Zernike functions 1/1

- Zernike functions
  - introduced by ***Frits Zernike***, a Nobel prize-winner physicist
  - ***to model*** *symmetries* and *aberrations* of ***optical systems*** (e.g., telescopes)
  - various schemes of *normalization* and *numbering*
  - an ***example*** of *Zernike functions*, namely Y_3^2, is shown below; its *index-pair* is shown as a red dot in the trapezoid-shaped *index-space*





Introduction – Zernike functions in ophthalmology 1/2

- Zernike functions and coefficients
  - nowadays, ***ophthalmologists*** are quite *familiar* with the *Zernike surfaces* that smoothly wave *over the unit disk*
  - they use these surfaces exactly in the way as was intended by Zernike, that is, to *describe* various *symmetries* and *aberrations* of an ***optical system***
  - in this case, of the ***human eye***
  - more precisely of the ***corneal surface*** – measured with some *corneal topographer*
  - of the refractive properties of the ***eyeball*** – measured with a *Shack-Hartmann wavefront-sensor*
  - the *description* is given in the form of *Zernike cofficients*

Introduction – Zernike functions in ophthalmology 2/2

- The optical aberrations
  - may cause serious ***acuity problems***, and
  - significant factors to be considered in planning of *sight-correcting operations*
  - wide range of ***statistical data*** concerning the *eyes* of various *groups of people* is available for the most important Zernike coefficients
  - difficult – or in certain cases impossible – to take high-resolution retinal images without *compensating the aberrations* of the eye, however, by compensating them ***high-resolution retinal imaging*** can be achieved

Introduction – Discrete orthogonal system of Zernike functions 1/1

- Utilizing discrete orthogonality
  - although, the corneal *Zernike coefficients* have always been ***obtained from*** measurements at *discrete corneal points*
  - via computations using ***some discretization*** of the *continuous Zernike functions*
  - the *developers* of these algorithms ***could not rely on*** the *discrete orthogonality* of Zernike functions
  - simply because ***no mesh of points ensuring discrete orthogonality was known***
  - the discrete orthogonality of Zernike functions was a ***target of considerable research*** for some time
  - ***only recently*** was a *mesh of points* ensuring discrete orthogonality of the Zernike functions found and *introduced by* Pap and Schipp

Continuous Zernike functions

– a ***surface*** over the unit disk can be *described* by a *two-variable function g(x, y)*

– the application of the *polar-transform* to variables *x* and *y* results in

$$x = \rho \cos \vartheta, \quad y = \rho \sin \vartheta,$$

where ρ and θ are the *radial and the azimuthal variables,* respectively, over the *unit disk*, i.e., where

$$0 \leq \rho \leq 1, \quad 0 \leq \vartheta \leq 2\pi.$$

– using ρ and θ, *g(x, y) can be transcribed* into the following form

$$G(\rho, \vartheta) := g(\rho \cos \vartheta, \rho \sin \vartheta).$$

Continuous Zernike functions 2/3

- the ***set of Zernike polynomials*** *of degree less than 2N* is as follows

$$Y_n^l(\rho,\vartheta) := \sqrt{2n+|l|+1} \cdot R_{|l|+2n}^{|l|}(\rho) \cdot e^{il\vartheta}$$
$$(l \in \mathbb{Z}, n \in \mathbb{N}, |l|+2n < 2N)$$

- the ***radial polynomial*** marked above *can be expressed* with *Jacobi polynomials* $P_k^{\alpha,\beta}$ in the following manner:

$$R_{|l|+2n}^{|l|}(\rho) = \rho^{|l|} \cdot P_n^{0,|l|}(2\rho^2 - 1).$$

- *some Zernike polynomials:*

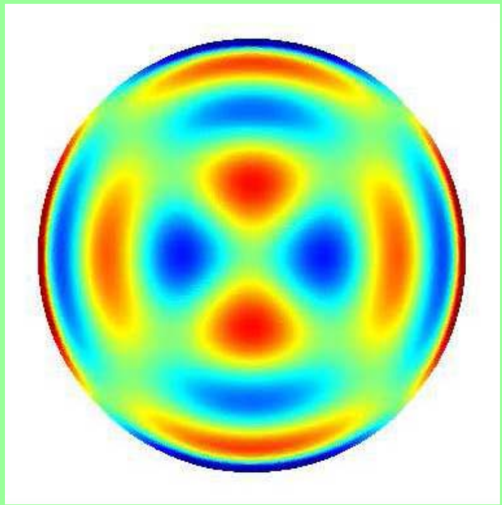
$$R_0^0 = 1,\ R_2^0 = 2\rho^2 - 1,\ R_4^0 = 6\rho^4 - 6\rho^2 + 1$$

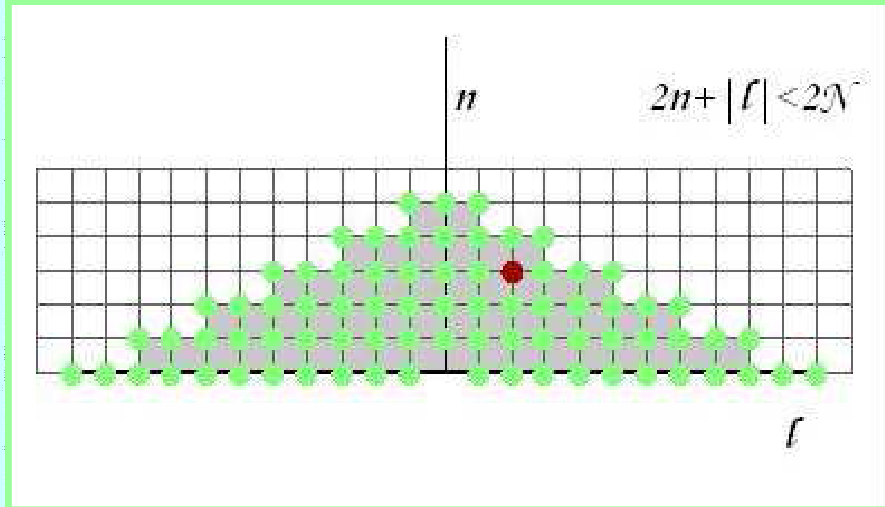
$$R_1^1 = \rho,\ R_3^1 = 3\rho^3 - 2\rho.$$

Continuous Zernike functions

– *an* ***example*** *of Zernike functions*, namely Y_3^2 , is shown in a pseudo-colour representation

– the ***index-space*** is shown for *N* = *6*, that is, for the *set of Zernike polynomials of degree less than 12*





Discretization of Zernike functions — Mesh ensuring the discrete orthogonality of Zernike functions 1/4

– the mesh, or the ***set of nodal points*** proven to *ensure* the *discrete orthogonality of Zernike functions* (over this mesh) is as follows:

$$X_N := \{z_{jk} := (\rho_k^N, \frac{2\pi j}{4N+1}) : k = 1, ..., N, j = 0, .., 4N\},$$

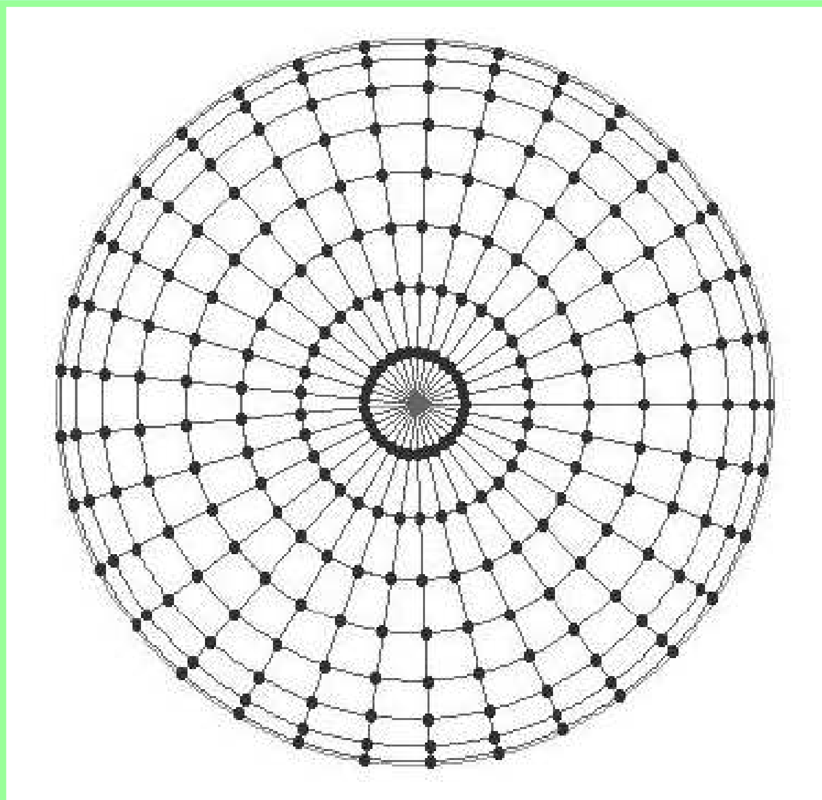
where

$$\rho_k^N := \sqrt{\frac{1+\lambda_k^N}{2}}, \quad k = 1, ..., N.$$

λ_k^N is the *k-th root* of the *Legendre polynomial* P_N of order N

Discretization of Zernike functions — Mesh ensuring the discrete orthogonality of Zernike functions 2/4

– ***set of nodal points*** X_8



Discretization of Zernike functions — Mesh ensuring the discrete orthogonality of Zernike functions 3/4

– by using the following ***discrete integral***

$$\int_{X_N} f(\rho,\phi)\,d\nu_N := \sum_{k=1}^{N}\sum_{j=0}^{4N} f\left(\rho_k^N, \frac{2\pi j}{4N+1}\right)\frac{A_k^N}{2(4N+1)}$$

the *discrete orthogonality* of the Zernike functions *can be proven*

– the marked *weights* are associated with the *discrete circular rings* in the mesh

– the ***discrete orthogonality relation*** is as follows

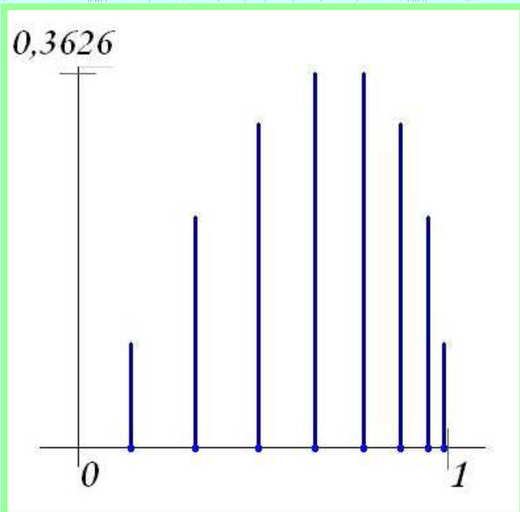
$$\int_X Y_n^m(\rho,\phi)\overline{Y_{n'}^{m'}(\rho,\phi)}\,d\nu_N = \delta_{nn'}\delta_{mm'}.$$

$$n + n' + |m| < 2N$$

$$n + n' + |m'| < 2N$$

Discretization of Zernike functions — Mesh ensuring the discrete orthogonality of Zernike functions 4/4

– the ***weights*** $A_1^8, \ldots, A_8^8$



Discretization of Zernike functions — Significance of the quadrature formulas 1/1

- ***quadrature formulas*** are known for some well-researched *continuous orthogonal polynomials* of one variable since *Gauss*'s time
- these are expressed in the following way:

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=1}^{N} f(\lambda_k^N) A_k^N .$$

- the ***integration of function*** *f(x) using its quadrature formula* is much *more precise* than a numerical integration using *over some arbitrary*, e.g., equidistant mesh
- in our case, that is, ***for the discretization of the radial Zernike polynomials*** the *N* roots of *Legendre-polynomials* P_N were used
- though, the *formula* for deriving the weights is not given here, it is ***exact*** *for every polynomial f* of order less than *2N*

The discrete Zernike coefficients

- the ***discrete Zernike coefficients*** *associated with* function *T(ρ, θ) can be calculated* with the following ***discrete integral***

$$C_n^m = \frac{1}{\pi} \int_{X_N} T(\rho, \phi) \overline{Y_n^m(\rho, \phi)} d\nu_N .$$

- if *T(ρ, θ)* happens to be an *arbitrary linear combination of Zernike functions* of degree less than *2N*, *then the above discrete integrals*

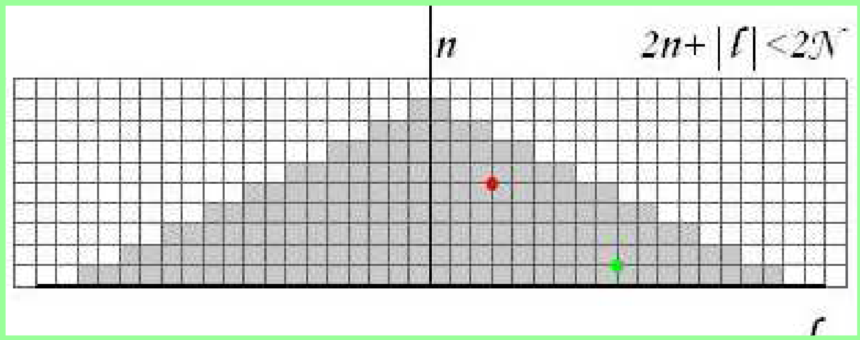
- for *n*'s and *m*'s satisfying the inequality *2n* + |*m*| < *2N*

- result in the *exact Zernike coefficients*, i.e., the ones that are calculated from the corresponding continuous integrals

The discrete Zernike coefficients — Program implementation 2/4

- a *program implementation* was developed for computing the discrete Zernike coefficients

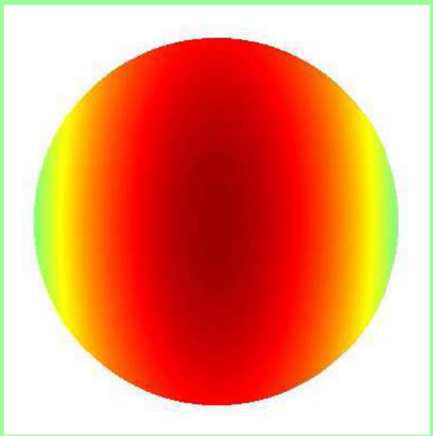
- with the developed program, the *precision of the discrete orthogonality* can be checked for the mesh of points

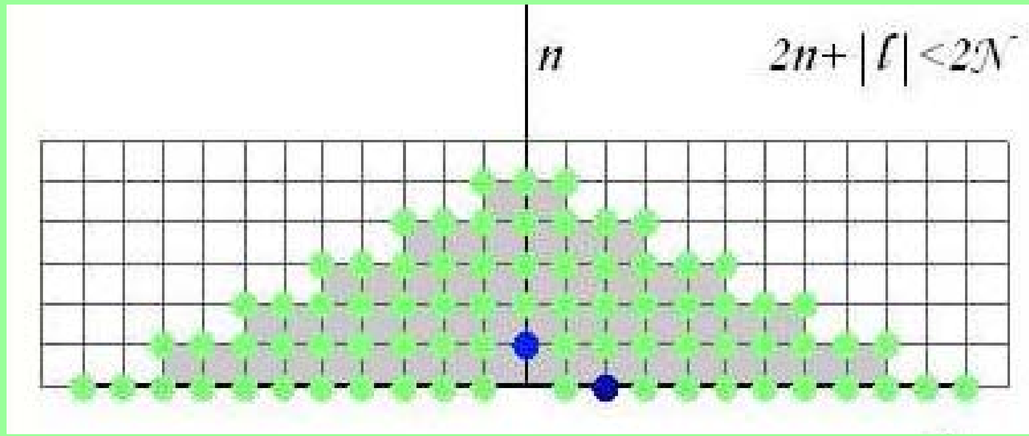


- the *precision of the discrete orthogonality* for the above index-set and for its corresponding mesh, for the two Zernike functions (with the marked indices) the *error* was $3.8 \cdot 10^{-18}$

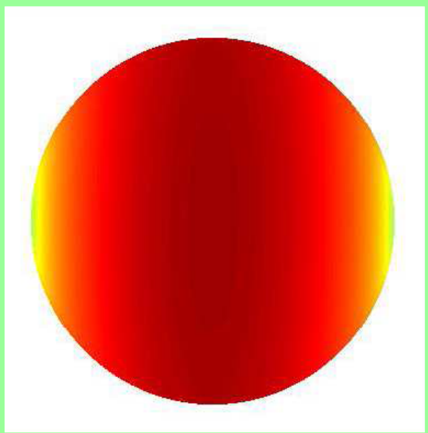
The discrete Zernike coefficients — Examples 3/4

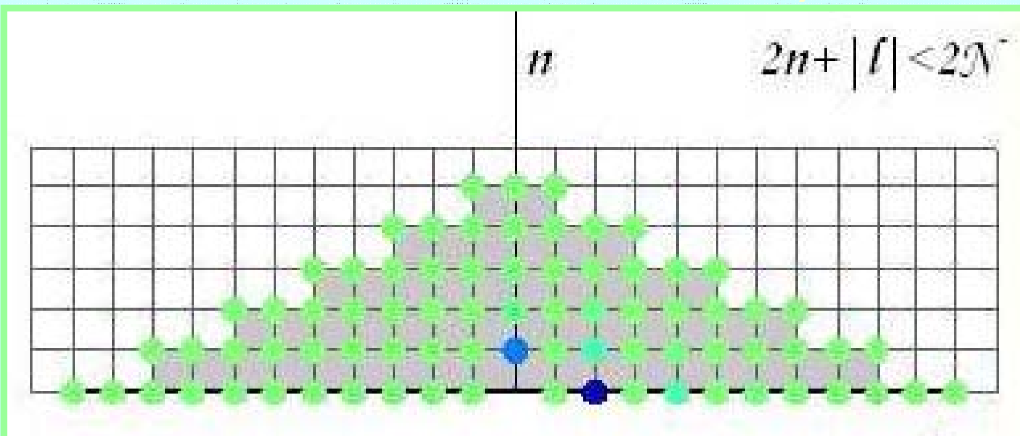
– the *input functions* were selected from the ***test surfaces for corneal topographers***





– *two sphero-cylindrical surfaces* and their Zernike coefficients, note that the lower one is more cylindrical





The discrete Zernike coefficients — Program implementation 3/4

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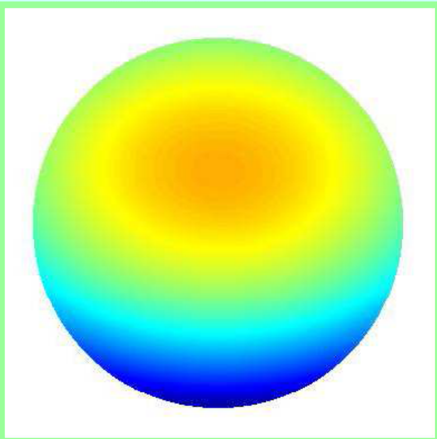

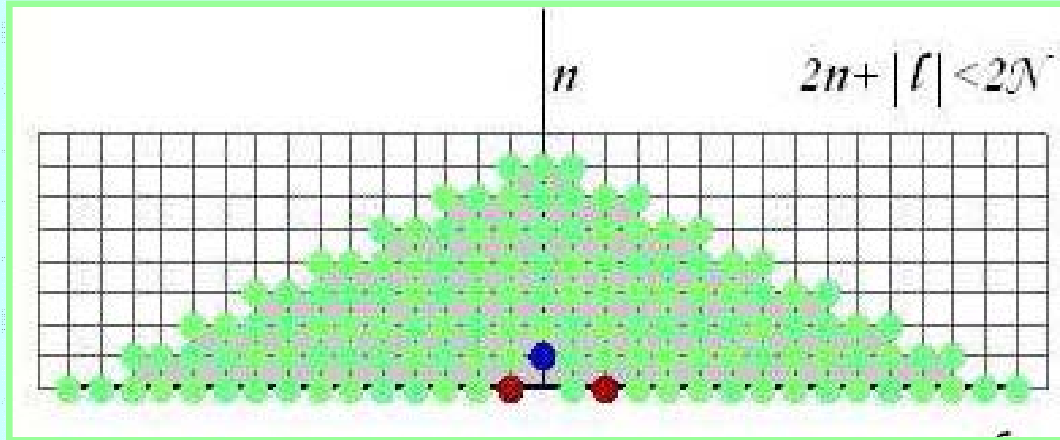

- a *surface modelling a deformed cornea,* called *keratoconus,* and its *active Zernike cofficients*

Conclusions and future work

- the ***discretization*** used in this paper was proposed by Pap and Schipp
- it has ***relevance*** *to the concrete application field*, but could *also benefit physicists and engineers* dealing with optical measurements and measurement devices
- however, *using this mesh as a measurement-pattern* in a reflective corneal topographer ***will not result in*** a sampling that ensures ***discrete orthogonality*** of the Zernike functions *as the corneal surfaces does not have a standard shape*
- ***to benefit in reflective corneal topography from the discretization*** used in this paper, the *optical system* – together with the *internal control mechanism* – of the topographer must ensure that the *sampling points on the corneal surface* are positioned *according to the mesh* with respect to the optical axis of the camera
- the above requirement is best achieved by some ***adaptive optical mechanism*** *and appropriate control*

References

Zernike, F., *Beugungstheorie des Schneiden-verfahrans und Seiner Verbesserten Form, der Phase-kontrast methode*, Physica, Vol. 1, pp. 1137-1144, 1934.

Szegő G., *Orthogonal polynomials*, 4th Edition, AMS, New York, 1981.

Wyant, J.C., Creath, K., *Basic wavefront aberration theory for optical metrology*, Applied Optics and Optical Engineering, Vol. 11, Academic Press, New York, 1992.

Corbett, M.C., Rosen E.S., O'Brart, D.P.S., *Corneal topography: principles and practice*, Bmj Publ. Group, London, UK, 1999.

References

Iskander, D.R., Collins, M.J., Davis, B., *Optimal modeling of corneal surfaces with Zernike polynomials*, IEEE Transactions on Biomedical Engineering, Vol. 48, No. 11, pp. 87-95, 2001.

Iskander, D.R., Morelande, M.R., Collins, M.J., Davis, B., *Modeling of corneal surfaces with radial polynomials,* IEEE Transctions on Biomedical Engineering, Vol. 49, No. 11, pp. 320-328, 2002.

Savarese, S. Chen, M. Perona P., *Second order local analysis for 3D reconstruction of specular surfaces*, IEEE First International Symposium on 3D Data Processing Visualisation and Transmission, 2002.

Pap, M., Schipp, F., *Discrete orthogonality of Zernike functions,* Mathematica Pannonica, Vol. 1, pp. 689-704, 2005.

References

Soumelidis, A., Fazekas, Z. Schipp, F., *Surface description for cornea topography using Modified Chebyshev-polynomials,* 16th IFAC World Congress, Prague, Czech Republic, pp. Fr-M19-TO/5, 2005.

Soumelidis, A., Csákány, B., *Specification of test cornea surfaces*, Project Report CORNEA-INT-2M02, MTA-SZTAKI, Budapest, Hungary, 2005.

Salmon, T. O., van de Pol, C., *Normal-eye Zernike coefficients and root-mean-square wavefront errors,* Journal of Cataract and Refractive Surgery, Vol. 32, No. 12, pp. 2064-2074, 2006.

References

Ling, N., Zhang, Y., Rao, X., Wang, C., Hu, Y., Jiang. W., Jiang, C., *Adaptive optical system for retina imaging approaches clinic applications,* Series Springer Proceedings in Physics, Springer Verlag, Berlin, Germany, pp. 305-315, 2006.

Soumelidis, A., Fazekas, Z., Schipp, F., Edelmayer, A., Németh, J., Csákány, B., *Development of a multi-camera corneal topographer using an embedded computing approach,* 1st Int. Conf. on Biomedical Electronics and Devices, Funchal, Madeira, Portugal, pp. 126-129, 2008.

Fazekas, Z., Soumelidis, A., Bódis-Szomorú, A., Schipp, F., *Specular surface reconstruction for multi-camera corneal topographer arrangements,* 30th Annual Int. IEEE EMBS Conference, Vancouver, Canada, pp. 2254-2257, 2008.