How to improve perimeter estimation by using grey levels

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Outline

1. Introduction
2. Perimeter estimation
   - Perimeter from a crisp digitization
   - Pixel coverage representations
   - Slopes and difference of column sums
   - Optimization to find γ_n
   - Local computations
   - Complete algorithm
   - Evaluation and Examples
3. Conclusion

Outline

To introduce myself...

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B.Sc. in Mathematics, University of Novi Sad
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...my little group...

Currently working with two Ph.D. students at the University of Novi Sad, in cooperation with Ass. Prof. Joakim Lindblad, CBA, Uppsala, Sweden:

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Vladimir Ćurić, M.Sc.

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Currently working with two Ph.D. students at the University of Novi Sad, in cooperation with Ass. Prof. Joakim Lindblad, CBA, Uppsala, Sweden:

Tibor Lukić, M.Sc.

problems of defuzzification and image regularization, where focus is in improved performance of the used algorithms due to incorporated SPG numerical optimization method.

Vladimir Ćurić, M.Sc.

problems related to measuring distance between sets (objects); the results are intended to be applied in image registration.

Activities during the last year

(June 2008 – June 2009)

Published

1. N. Sladoje, J. Lindblad: High Precision Boundary Length Estimation by Utilizing Gray-Level Information. IEEE Trans. on Pattern Analysis and Machine Intelligence, Vol. 31, No. 2, pp. 357-363 (2009).
2. T. Lukić, N. Sladoje, J. Lindblad: Deterministic Defuzzification based on Spectral Projected Gradient Optimization, Proc. of DAGM08, Lecture Notes in Computer Science, Springer, Vol. 5096, pp 476-485 (2008).
3. J. Lindblad, N. Sladoje, V. Ćurić, H. Sarve, C.B. Johansson, G. Borgefors. Improved quantification of bone remodelling by utilizing fuzzy based segmentation. Lecture Notes in Computer Science, Springer, Vol. 5575, pp ?-? (2009).
4. A. Tanács, C. Domokos, N. Sladoje, J. Lindblad, Z. Kato. Recovering affine deformations of fuzzy shapes. Lecture Notes in Computer Science, Springer, Vol. 5575, pp 735-744 (2009).

Accepted

5. N. Sladoje, J. Lindblad. Pixel coverage segmentation for improved feature estimation. Lecture Notes in Computer Science, Springer, 2009.
6. J. Lindblad, V. Ćurić, N. Sladoje. On set distances and their application to image registration. 6th IEEE Intern. Symp. on Image and Signal Processing and Analysis, ISPA 2009.

...and to introduce the topic

- The task of image analysis is to extract relevant information from images.
- Images contain discrete representations of real continuous objects.
- Our aim is usually to obtain information about continuous real objects, having available their discrete representations.
- Different numerical descriptors, such as area, perimeter, moments of the objects are often of interest, for the tasks of shape analysis, classification, etc.

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- Images contain discrete representations of real continuous objects.
- Our aim is usually to obtain information about continuous real objects, having available their discrete representations.
- Different numerical descriptors, such as area, perimeter, moments of the objects are often of interest, for the tasks of shape analysis, classification, etc.
- Accurate and precise **perimeter estimates** of real objects, based on their discrete representations, have been of interest for more than forty years; the problem still attracts attention.

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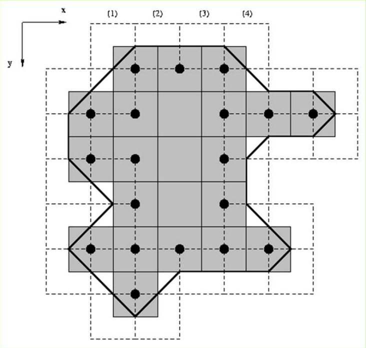
Perimeter estimation

Formulation of the problem:

Having a **discrete representation** of a real object, digitized in an integer grid, **estimate** its perimeter (length of its border) with as small error as possible.

We wish to obtain as correct feature values as possible - **accuracy**, and that repeated measurements provide similar results - **precision**.



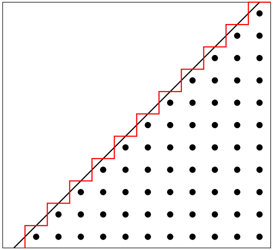
One approach

Local polygonalization
Approximate the object perimeter with the perimeter of a locally defined polygon.

Direct use of the perimeter of the polygon gives, on average, an **overestimate**.

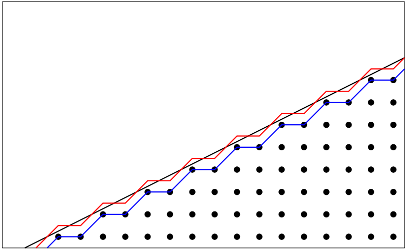
How to assign local step lengths

Using 4 edge directions.

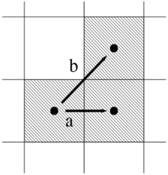


Digital edge $\sqrt{2}$ times longer than true edge.

Using 8 edge directions.



Edge 1.08 times longer than true edge.



Freeman 1970:
$a = 1, \quad b = \sqrt{2}$

Using $a = 1$, $b = \sqrt{2}$ lead to an **overestimate**.

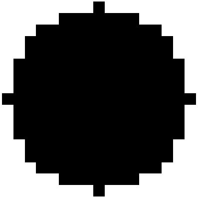
Error minimization

Verwer 1991

- Decide what error to minimize
  - The mean square error (MSE) minimization leads to estimators that, in average, perform well for lines of all directions.
  - The maximal error minimization leads to estimator with a better “controllable” error.
- Compute **optimal step lengths** to minimize the chosen error measure when estimating the length of straight segments of arbitrary direction.

- To minimize MSE: $a = 0.9481$ and $b = 1.3408$.
  Root Mean Square (RMS) Error is 2.33%.
- To minimize MaxErr: $a = 0.9604$ and $b = 1.3583$.
  Maximal Error is 3.95%.
- **The error does not decrease with increasing resolution**

Grey-level images



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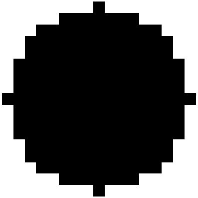

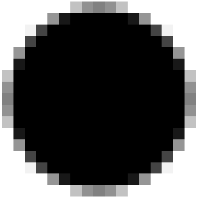

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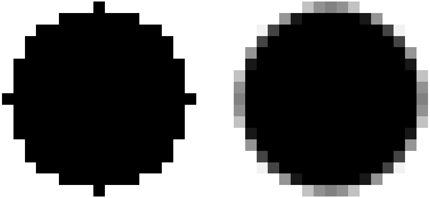

Background and previous work

- “Grey levels can improve the performance of binary image digitizers” - N. Kiryati and A. Bruckstein, 1991.
- Work of Eberly and Lancaster, 1991, and Verbeek and van Vliet, 1993, showed attempts to use grey-level information for length estimation, but appeared to be surprisingly “non-inspirative” for the scientific ancestors.

Pixel coverage digitization

Let a square grid in 2D be given. The Voronoi region associated to a grid point $(i,j) \in \mathbb{Z}^2$ is called pixel $p_{(i,j)}$.

Definition (non-quantized case)

For a given continuous object $S \subset \mathbb{R}^2$, inscribed into an integer grid with pixels $p_{(i,j)}$, the *pixel coverage digitization* of S is

$$\mathcal{D}(S) = \left\{ \left((i,j), \frac{A(p_{(i,j)} \cap S)}{A(p_{(i,j)})} \right) \,\middle|\, (i,j) \in \mathbb{Z}^2 \right\} \;,$$

where $A(X)$ denotes the area of a set X.

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Digital images $\rightarrow$ Quantized grey values

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Digital images $\rightarrow$ Quantized grey values

Definition (n-level quantized case)

$$\mathcal{D}^n(S) = \left\{ \left((i,j), \frac{1}{n} \left\lfloor n \frac{A(p_{(i,j)} \cap S)}{A(p_{(i,j)})} + \frac{1}{2} \right\rfloor \right) \,\middle|\, (i,j) \in \mathbb{Z}^2 \right\} ,$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

The straight edge of a halfplane

Discrete, grey-scale, non-quantized

Observe a halfplane $H = \{(x,y) \mid y(x) \leq kx + m,\ k,m \in [0,1]\}$, over an interval $x \in [0,N]$, $N \in \mathbb{Z}^+$.
Let I be the non-quantized pixel coverage digitization $I = \mathcal{D}(H)$ ($\Delta x = \Delta y = h = 1$ by definition.)

Then it holds that

$$y(i) = \sum_{j \geq 0} I(i,j) - 0.5$$

$$k(i) = y(i+1) - y(i) = k$$

$$l = \sqrt{N^2 + (kN)^2} = \sum_{i=0}^{N-1} \sqrt{1 + k(i)^2}$$

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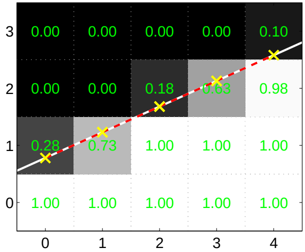
The length of the edge segment l is "estimated" with **no error**.

Non-quantized example

$l = 4.39$



s_c:	1.28		1.73		2.18		2.63		3.08
d_c:		0.45		0.45		0.45		0.45	
l_c:		1.10		1.10		1.10		1.10	

Figure: Example illustrating edge length estimation based on the difference d_c of column sums s_c for a segment ($N = 4$) of a halfplane edge given by $y \leq 0.45x + 0.78$.

$$s_c = \sum_{j \geq 0} I(c,j)\ ,\ d_c = s_{c+1} - s_c\ ,\ l_c = \sqrt{1 + d_c^2}$$

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Let I be the **quantized** pixel coverage digitization $I = \mathcal{D}^n(H)$

Then

$$\tilde{l} = \sum_{c=0}^{N-1} \sqrt{1 + d_c^2}$$

provides an estimate of the edge length l.

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Discrete, grey-scale, quantized

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Then

$$\tilde{l} = \sum_{c=0}^{N-1} \sqrt{1 + d_c^2}$$

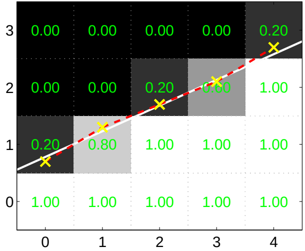
provides an estimate of the edge length l.

However, this is in general an overestimate (zig-zag steps).
Scaling the estimate with an optimally chosen factor $\gamma_n < 1$, gives an estimate with a minimal error.

$$\hat{l} = \sum_{c=0}^{N-1} \gamma_n \sqrt{1 + d_c^2}$$

Quantized example

$$\hat{l} = \gamma_5 * 4.49$$



$\tilde{s}_c$:	1.20		1.80		2.20		2.60		3.20
$\tilde{d}_c$:		0.60		0.40		0.40		0.60	
$\tilde{l}_c$:		1.17		1.08		1.08		1.17	

Figure: Example illustrating edge length estimation based on the difference d_c of column sums s_c for a segment ($N = 4$) of a halfplane edge given by $y \leq 0.45x + 0.78$.

$$s_c = \sum_{j \geq 0} I(c, j) \ , \ d_c = s_{c+1} - s_c \ , \ l_c = \sqrt{1 + d_c^2}$$

Minimization of the maximal relative error

An edge is a linear combination of local steps

The edge segment $\mathbf{l} = (N, kN)$ can be expressed as a linear combination of two of the vectors, $\mathbf{S}_i = (1, \frac{i}{n})$, $\mathbf{S}_j = (1, \frac{j}{n})$, $i, j \in \{0, 1, \ldots, n\}$, having slopes $k_i = \frac{i}{n}$, $k_j = \frac{j}{n}$ such that $k_i \leq k \leq k_j$.
Its length on the interval $[0, N]$ can be estimated by

$$\hat{l} = \gamma_n \left(\frac{(j - nk)N}{j - i} S_i + \frac{(nk - i)N}{j - i} S_j \right) \text{ , where } S_i = \sqrt{1 + (\tfrac{i}{n})^2} \text{ .}$$

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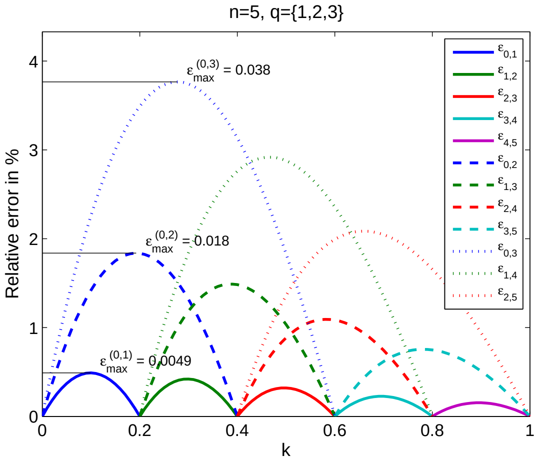
Relative error of the length estimation

The relative error of the length estimation of the line segment with slope k, such that $k \in [\frac{i}{n}, \frac{j}{n})$:

$$\varepsilon_{i,j}(k) = \frac{\hat{l} - l}{l} = \gamma_n \frac{(j - nk)S_i + (nk - i)S_j}{(j - i)\sqrt{1 + k^2}} - 1 \ .$$

Minimization of the maximal relative error

A plot of the absolute value of the relative error $\varepsilon_{i,j}(k)$ for $n = 5$:



Minimization of the maximal relative error

Result [Sladoje and Lindblad, PAMI 2009]

The maximal error is minimized for

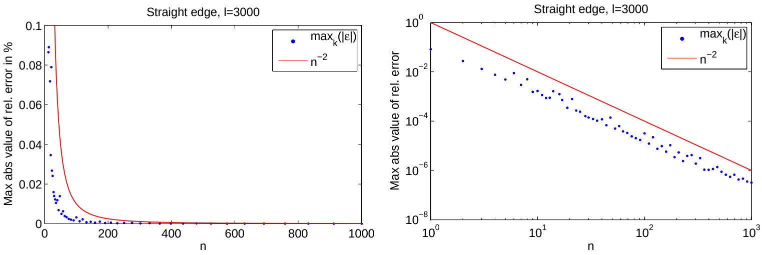
$$\gamma_n^q = \frac{2q}{q + \sqrt{(\sqrt{n^2+q^2} - n)^2 + q^2}}, \text{ where } q = j - i\ .$$

The maximal error is $|\varepsilon| = 1 - \gamma_n^q$.
Quantization leads to $q > 1$. In 2D it holds that $q \leq 3$.

Asymptotic behaviour

Observing the estimation error as a function of the number of grey-levels n, we conclude that

$$|\varepsilon_n| = \mathcal{O}\left(\frac{1}{n^2}\right) .$$

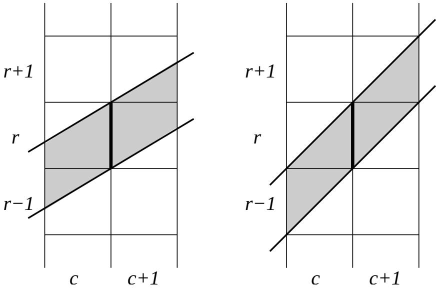


(a) lin-lin scale (b) log-log scale

Figure: Asymptotic behaviour of the maximal error for straight edge length estimation using $\gamma_n = \gamma_n^1$; theoretical (line) and empirical (points) results.

Assignment of a length to a segment

For lines of a slope $k \in [0, 1]$, each value d_c depends on at most six pixels, located in a 3×2 rectangle:



(a) $k = 0.6$ (b) $k = 1$

Figure: Regions where lines $y = kx + m$ with k, m such that $r - \frac{1}{2} \leq u = k(c + \frac{1}{2}) + m \leq r + \frac{1}{2}$, intersect a 3×2 configuration.

Assignment of a length to a segment - local conditions

Analytically defined half-plane

A condition to be checked for a 3×2 neighbourhood of a pixel $p_{(c,r)}$ in continuous case is

$$r - \frac{1}{2} \leq u = k(c + \frac{1}{2}) + m \leq r + \frac{1}{2},$$

for a line $y = kx + m$

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Locally observed discretized half-plane

In a discrete case, $u = k(c + \frac{1}{2}) + m$ is **e**stimated by

$$\tilde{u} = r - \frac{3}{2} + \frac{1}{2} \sum_{i=1}^{6} \tilde{p}_i$$

and the same condition

$$r - \frac{1}{2} \leq \tilde{u} \leq r + \frac{1}{2}$$

is used.

Local contributions calculations

Local contributions to the length of a segment

A local length assigned to a 3×2 neighbourhood of a pixel $p_{(c,r)}$ is

$$\tilde{l}^{D}_{(c,r)} = \begin{cases} \sqrt{1 + d^2_{(c,r)}}\,, & \tilde{u} \in \left(r - \frac{1}{2}, r + \frac{1}{2}\right) \\ \frac{1}{2}\sqrt{1 + d^2_{(c,r)}}\,, & \tilde{u} = r \pm \frac{1}{2} \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Isometries of the plane - cases when $|k| \notin [0, 1]$

- If $|k| \notin [0, 1]$, we need 2×3 configuration to estimate the slope; when we exchange roles of the axes, we can apply the same algorithms as for the former case.
- Instead of changing size of configuration depending on k, we suggest to use 3×3 configurations in all cases.
- In any 3×3 configuration we (would like to) use the algorithms described for case $|k| \in [0, 1]$.
- To be able to do that, we need to apply isometric transformations to the 3×3 configurations correspondent to $|k| \notin [0, 1]$.

Conditions for selection of isometries

Analytically defined half-plane

In a continuous case, for a known **analytically defined** half-plane, it holds:

- If $y \geq kx + m$, then symmetry w.r.t. the line $y = r$ should be applied;
- If $k < 0$, then symmetry w.r.t. the line $x = c$ should be applied;
- Finally, if $|k| > 1$, symmetry w.r.t. the line $x + y = r + c$ should be applied.

Conditions for selection of isometries

Locally observed dicretized half-planes

We can use **quantized** pixel values from a 3×3 neighbourhood in the following set of criteria:

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Let

$$
\begin{aligned}
\tilde{\alpha} &= \tilde{p}_7 + \tilde{p}_8 + \tilde{p}_9 - \tilde{p}_1 - \tilde{p}_2 - \tilde{p}_3 \ , \\
\tilde{\beta} &= \tilde{p}_3 + \tilde{p}_6 + \tilde{p}_9 - \tilde{p}_1 - \tilde{p}_4 - \tilde{p}_7 \ , \\
\tilde{\delta} &= \tilde{p}_4 + \tilde{p}_7 + \tilde{p}_8 - \tilde{p}_2 - \tilde{p}_3 - \tilde{p}_6 \ .
\end{aligned}
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\end{aligned}
$$

If $\tilde{\alpha} > 0$ then $H : y \leq kx + m$. (all fine)
If $\tilde{\alpha} < 0$ then $H : y \geq kx + m$. Exchange the first and the third row.
If $\tilde{\alpha} = 0$ then all rows are equal. Leave as it is.

If $\tilde{\beta} > 0$ then $k > 0$. (all fine)
If $\tilde{\beta} < 0$ then $k < 0$. Exchange the first and the third column.
If $\tilde{\beta} = 0$ then all columns are equal. Leave as it is.

If $\tilde{\delta} > 0$ then $k < 1$. (all fine)
If $\tilde{\delta} < 0$ then $k > 1$. The symmetry w.r.t. $x + y = r + c$ is to be performed.
If $\tilde{\delta} = 0$ then “corners” are equal. Leave as it is.

Algorithm

Input: Pixel coverage values $\tilde{p}_i,\ i = 1, \ldots, 9$, from a 3×3 neighbourhood $T_{(c,r)}$.
Output: Local edge length $\hat{l}^T_{(c,r)}$ for the given 3×3 configuration.

---

if $\tilde{p}_7 + \tilde{p}_8 + \tilde{p}_9 < \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3$ /* $y \geq kx + m$ */
  swap$(\tilde{p}_1, \tilde{p}_7)$
  swap$(\tilde{p}_2, \tilde{p}_8)$
  swap$(\tilde{p}_3, \tilde{p}_9)$
endif

if $\tilde{p}_3 + \tilde{p}_6 + \tilde{p}_9 < \tilde{p}_1 + \tilde{p}_4 + \tilde{p}_7$ /* $k < 0$ */
  swap$(\tilde{p}_1, \tilde{p}_3)$
  swap$(\tilde{p}_4, \tilde{p}_6)$
  swap$(\tilde{p}_7, \tilde{p}_9)$
endif

if $\tilde{p}_4 + \tilde{p}_7 + \tilde{p}_8 < \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_6$ /* $k > 1$ */
  swap$(\tilde{p}_2, \tilde{p}_4)$
  swap$(\tilde{p}_3, \tilde{p}_7)$
  swap$(\tilde{p}_6, \tilde{p}_8)$
endif

$\tilde{s}_1 = \tilde{p}_1 + \tilde{p}_4 + \tilde{p}_7$
$\tilde{s}_2 = \tilde{p}_2 + \tilde{p}_5 + \tilde{p}_8$
$\tilde{s}_3 = \tilde{p}_3 + \tilde{p}_6 + \tilde{p}_9$

$\tilde{u}_l = (\tilde{s}_1 + \tilde{s}_2)/2$
$\tilde{u}_r = (\tilde{s}_2 + \tilde{s}_3)/2$

if $1 \leq \tilde{u}_l < 2$
  $\tilde{d}_l = \tilde{s}_2 - \tilde{s}_1$
  $\hat{l}_l = \frac{\gamma_n}{2}\sqrt{1 + \tilde{d}_l^2}$
else
  $\hat{l}_l = 0$
endif

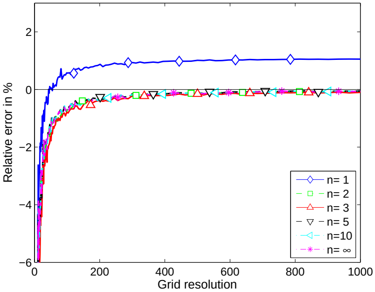
if $1 < \tilde{u}_r \leq 2$
  $\tilde{d}_r = \tilde{s}_3 - \tilde{s}_2$
  $\hat{l}_r = \frac{\gamma_n}{2}\sqrt{1 + \tilde{d}_r^2}$
else
  $\hat{l}_r = 0$
endif

$\hat{l}^T_{(c,r)} = \hat{l}_l + \hat{l}_r$

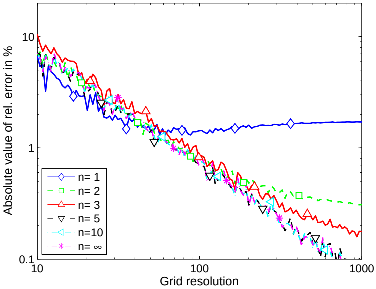
Only **integer arithmetics** used locally (fast, exact).
Only **local** information is used (fast, stable, parallelizable).

Estimation errors

Trade-off between spatial and grey-level resolution



(a) lin-lin scale



(b) log-log scale

Figure: Relative errors in percent for test shapes digitized at increasing resolution for 5 different quantization levels and non-quantized ($n = \infty$).

Outline

Instead of Conclusions - Pixel Coverage Segmentation

To use the method we need pixel coverage images.

We have proposed three segmentation methods which provide (approximate) pixel coverage images:

1. A method based on mathematical morphology and a dual thresholding scheme.
   - N. Sladoje and J. Lindblad. High Precision Boundary Length Estimation by Utilizing Gray-Level Information. IEEE Trans. on PAMI, Vol. 31, No. 2, pp. 357-363, 2009.
2. A method providing local sub-pixel classification extending any existing crisp segmentation.
   - N. Sladoje and J. Lindblad. Pixel coverage segmentation for improved feature estimation. Accepted for LNCS on ICIAP 2009.
3. Direct assignment of area coverage values from a continuous segmentation model.
   - A. Tanács, C. Domokos, N. Sladoje, J. Lindblad, and Z. Kato. Recovering affine deformations of fuzzy shapes. Proc. of SCIA, LNCS, Vol. 5575, pp. 735-744, 2009.