# An Axiomatic Framework for Finite Automata

## Szeged, September 2013

Zoltán Ésik

## Aims

• To show that the validity of several constructions in automata theory only depends on certain simple **equational properties of fixed operations**.

• To define and develop the basic theory of **Conway** and **iteration semirings**.

•To show the usefulness of these algebraic structures for automata theory by:

• showing that **Kleene's theorem** only depends on the Conway semiring identities, and

• providing **complete axiomatizations** of the equational theory of the semirings of **(regular) languages** and **(rational) power series**, and

• relating iteration semirings to **complete** and **continuous semirings**, and **inductive *-semirings** and **Kleene algebras**.

## Semirings

**Definition** A **semiring**

$$S = (S, +, \cdot, 0, 1)$$

- $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid.
- $$\begin{aligned} a(b+c) &= ab + ac \\ (b+c)a &= ba + ca \\ 0a &= a0 = 0. \end{aligned}$$

**Idempotent semiring**: $a + a = a$
**Commutative semiring**: $ab = ba$

A morphism of semirings preserves the operations and the constants.

## Semirings

### Examples

- The semiring $\mathbb{N}$ of nonnegative integers.
- The boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$
- The semiring $\mathbb{N}_\infty$ with underlying set $\mathbb{N} \cup \{\infty\}$
  $\infty + x = x + \infty = \infty$, $\infty y = y \infty = \infty$, $x, y \in \mathbb{N}_\infty$, $y \neq 0$
- The language semiring $P(A^*) = (P(A^*), \cup, \cdot, \emptyset, \{\epsilon\})$.
- The semiring $P(M)$ of all subsets of a monoid $M$.
- The semiring $\mathbf{Rel}(A) = (\mathbf{Rel}(A), \cup, \circ, \emptyset, \mathbf{Id})$ of binary relations.
- Polynomial semirings $S\langle A^* \rangle$.
- The tropical semirings
  $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$
- Any ring (thus any field) and any bounded distributive lattice.

## Why semirings?

A **finite automaton** over a semiring $S$ is a finite directed graph whose edges are labeled in the semiring $S$. (Some vertices may also carry a label in $S$.)

The **behavior** of the automaton is an element of $S$.

### Examples

- Classical finite automata.
- Weighted finite automata.
- Iterative programs.

Automata can be represented by **matrices**.

## Matrix semirings

When $S$ is a semiring and $n, m \geq 0$, we denote by $S^{n\times m}$ the set of all $n \times m$ matrices over $S$.

**Definition** Suppose that $S$ is a semiring and $n, m, p \geq 0$. For any matrices $A, B \in S^{n\times m}$, we define $A + B \in S^{n\times m}$:

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

And if $A \in S^{n\times m}$ and $B \in S^{m\times p}$, then we define $AB \in S^{n\times p}$ by

$$(AB)_{ij} \;=\; \sum_{k=1}^{m} A_{ik}B_{kj}.$$

The **zero matrix** $0_{mn} \in S^{m\times n}$ has all 0 entries. The **unit matrix** $E_n \in S^{n\times n}$ has 1's on the diagonal and 0's elsewhere.

## Matrix semirings

**Proposition** When $S$ is a semiring and $n \geq 0$,

$$S^{n\times n} = (S^{n\times n}, +, \cdot, 0, E_n)$$

is a semiring.

**Definition** Given a relation $\rho$ from the set of the first $n$ positive integers to the set of the first $m$ positive integers, there is an associated zero-one matrix, also denoted $\rho$ with $\rho_{ij} = 1$ iff $i\rho j$. We call this matrix a **relational matrix**, or a **functional matrix**, when $\rho$ is a function. A **permutation matrix** is a matrix associated with a permutation.

## Polynomial and power series semirings

**Definition** Given a semiring $S$ and a set $A$, a **(formal) power series** over $S$ and $A$:

$$f : A^* \to S \quad \text{or} \quad f = \sum_{u \in A^*} (f, u)u,$$

where $(f, u) = f(u)$ for all words $u$. **Support** of $f$: $\text{supp}(f) = \{u \in A^* : (f, u) \neq 0\}$. A **polynomial** is a series whose support is finite.

The sum of two series is defined pointwise. The product of two series $f, g$ is given by:

$$\begin{aligned}(fg, u) &= \sum_{u=xy} (f, x)(g, y) \quad \text{i.e.,} \\ (fg)(u) &= \sum_{u=xy} f(x)g(y), \quad u \in A^*.\end{aligned}$$

## Polynomial and power series semirings

The series 0 and the series 1 are given by

- $(0, u) = 0$, for all $u \in A^*$,
- $(1, \epsilon) = 1$ and $(1, u) = 0$, for all $u \in A^+$, where $\epsilon$ denotes the empty word.

We may embed $S$ and $A^*$ into $S\langle A^* \rangle$ in a natural way.

$$(s, u) = \begin{cases} s & \text{if } u = \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$(v, u) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}$$

## Polynomial and power series semirings

**Example** Each series in $\mathbb{B}\langle\langle A^*\rangle\rangle$ may be identified with a subset of $A^*$. $\mathbb{B}\langle\langle A^*\rangle\rangle$ is isomorphic to $P(A^*)$. $\mathbb{B}\langle A^*\rangle$ is isomorphic to $P_f(A^*)$, the semiring of finite subsets of $A^*$.

**Proposition** For every set $A$ and semiring $S$, $S\langle\langle A^*\rangle\rangle$ is a semiring containing $S\langle A^*\rangle$ as a subsemiring.

## Polynomial and power series semirings

**Theorem** The semiring $\mathbb{N}\langle A^*\rangle$ is the free semiring, freely generated by the set $A$.

Given any $h : A \to S'$, where $S'$ is a semiring, there is a unique way to extend $h$ to a semiring morphism $h^\sharp : \mathbb{N}\langle A^*\rangle \to S'$. First extend $h$ to a monoid morphism $\overline{h} : A^* \to S'$, then let

$$sh^\sharp = \sum_{u \in \mathsf{supp}(s)} (s, u)(u\overline{h})$$

for all $s \in \mathbb{N}\langle A^*\rangle$.

**Theorem** The semiring $\mathbb{B}\langle A^*\rangle$ (or $P_f(A^*)$) is the free idempotent semiring, freely generated by the set $A$.

## Polynomial and power series semirings

**Theorem** Given any semiring $S'$ and functions $h_S : S \to S'$ and $h : A \to S'$ such that $h_S$ is semiring morphism and each $sh_S$ commutes with any $ah$, there is a unique semiring morphism $h^\sharp : S\langle A^*\rangle \to S'$ extending both $h_S$ and $h$.

## Conway semirings

**Definition** A **$^*$-semiring** is a semiring $S$ equipped with a unary **star operation** $^* : S \to S$. Morphisms of $^*$-semirings are semiring morphisms preserving the star operation.

**Definition** A **Conway semiring** is a $^*$-semiring $S$ which satisfies the **product star** and **sum star** identities, i.e.,

$$\begin{aligned}(ab)^* &= a(ba)^*b + 1 \\ (a+b)^* &= a^*(ba^*)^*, \quad a, b \in S.\end{aligned}$$

## Conway semirings

**Proposition** The following identities hold in Conway semirings:

$$\begin{aligned} a^* &= aa^* + 1 \\ a^* &= a^*a + 1 \\ 0^* &= 1 \\ aa^* &= a^*a \\ (ab)^*a &= a(ba)^* \\ (a+b)^* &= (a^*b)^*a^* \end{aligned}$$

The first identity is the **star fixed point identity**. In any Conway semiring $S$ we define $a^+ = aa^* = a^*a$.

## Conway semirings

### Examples

- $\mathbb{B}$, $\mathbb{N}_\infty$

  $\mathbb{B}$: $0^* = 1^* = 1$, $\quad \mathbb{N}_\infty$: $0^* = 1$, $x^* = \infty$, $x \neq 0$
- $P(A^*)$

  $L^* = \{u_1 \ldots u_n : u_i \in L,\ n \geq 0\}$
- $\mathbb{T}$

  $x^* = 0$, $x \in \mathbb{N} \cup \{\infty\}$
- $\mathbf{Rel}(A)$

  $R^*$ is the reflexive-transitive closure of $R$

## Conway semirings

When $S$ is a $^*$-semiring, we turn each matrix semiring $S^{n\times n}$ into a $^*$-semiring. When $n = 0$, the definition of star is clear. When $n = 1$, we use the star operation of $S$. Suppose that $n > 1$ and let $m = n - 1$. We define:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $A \in S^{m\times m}$, $B \in S^{m\times 1}$, $C \in S^{1\times m}$, and $D \in S^{1\times 1}$, and where

$$\begin{array}{ll} \alpha = A^*B\delta CA^* + A^* & \beta = A^*B\delta \\ \gamma = \delta CA^* & \delta = (D + CA^*B)^*. \end{array}$$

## Conway semirings

**Theorem** (Conway, Krob, Bloom-Ésik) Suppose that $S$ is a Conway semiring. Then, by the above definition, so is each $S^{n\times n}$, and the matrix star formula holds for *all* possible decompositions of a matrix into four parts such that $A$ and $D$ are square matrices of any dimension. Moreover, the **star permutation identity** holds:

$$(\pi A \pi^T)^* = \pi A^* \pi^T$$

where $A \in S^{n\times n}$ and $\pi$ is an $n \times n$ permutation matrix with transpose (inverse) $\pi^T$.

## Conway semirings

**Note** The star permutation identity can be rephrased as the implication:

$$A\pi = \pi B \quad \Rightarrow \quad A^*\pi = \pi B^*$$

where $A, B$ are $n \times n$ and $\pi$ is an $n \times n$ **permutation** matrix. The identity $(AB)^* = E_n + A(BA)^*B$ holds for all matrices $A \in S^{n\times m}$ and $B \in S^{m\times n}$.

## Conway semirings

**Proposition** The following identities hold for matrices over a Conway semiring:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} (A+BD^*C)^* & (A+BD^*C)^*BD^* \\ (D+CA^*B)CA^* & (D+CA^*B)^* \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^* = \begin{pmatrix} A^* & A^*BD^* \\ 0 & D^* \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^* = \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}$$

## Conway semirings

Suppose that $S$ is a Conway semiring and $A$ is a set. Then we define a star operation on $S\langle\langle A^*\rangle\rangle$. Given $s \in S\langle\langle A^*\rangle\rangle$, let $s_0 = (s, \epsilon)$. Then for any word $u$ we define

$$(s^*, u) = \sum_{u=u_1\cdots u_n,\ u_i \in A^+} s_0^*(s, u_1)s_0^* \cdots s_0^*(s, u_n)s_0^*$$

**Theorem** (Bloom-Ésik) If $S$ is a Conway semiring, then so is any $S\langle\langle A^*\rangle\rangle$.

## Kleene's theorem in Conway semirings

Suppose that $S$ is a Conway semiring, $S_0$ is a sub Conway semiring of $S$, and $\Sigma \subseteq S$. Let $S_0\langle\Sigma\rangle$ denote the collection of all finite linear combinations over $\Sigma$ with coefficients in $S_0$.

**Definition** An **automaton** over $(S_0, \Sigma)$ is a triplet $\mathbf{A} = (\alpha, A, \beta)$, where $\alpha \in S_0^{1\times n}$, $A \in (S_0\langle\Sigma\rangle)^{n\times n}$, $\beta \in S_0^{n\times 1}$, called the **initial vector**, the **transition matrix** and the **final vector**. The **behavior** of $\mathbf{A}$ is:

$$|\mathbf{A}| = \alpha A^* \beta.$$

**Definition** We call $s \in S$ **recognizable** over $(S_0, \Sigma)$ if $s$ is the behavior of some automaton over $(S_0, \Sigma)$. **Notation:** $\mathbf{Rec}_S(S_0, \Sigma)$.

## Kleene's theorem in Conway semirings

**Examples**

1. Let $S = \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$, $S_0 = \mathbb{B}$, $\Sigma$ a finite set. Then an automaton over $(S_0, \Sigma)$ is an ordinary nondeterministic automaton, and its behavior is the characteristic series of the language accepted by the nondeterministic automaton.

2. Let $S_0$ be the semiring $\mathbb{N}_\infty$ and let $\Sigma$ be a finite set. Consider the semiring $S_0\langle\langle\Sigma^*\rangle\rangle$. Then an automaton $\mathbf{A} = (\alpha, A, \beta)$ over $(S_0, \Sigma)$ is a weighted automaton over $\Sigma$ with weights in $\mathbb{N}_\infty$ and the behavior is given for words $u = a_1 \ldots a_n$ by

$$(|\mathbf{A}|, u) = \sum_{i, i_1, \ldots, i_{n-1}, j} \alpha_i (A_{i,i_1}, a_1) \cdots (A_{i_{n-1},j}, a_n) \beta_j.$$

   Thus a series is recognizable over $(S_0, \Sigma)$ iff it is recognizable by a weighted automaton.

## Kleene's theorem in Conway semirings

**Definition** Let $S$ be a Conway semiring, $S_0$ a sub Conway semiring of $S$, and $\Sigma \subseteq S$. We call $s \in S$ **rational** over $(S_0, \Sigma)$ if $s$ is contained in the sub Conway semiring generated by $S_0 \cup \Sigma$.
**Notation:** $\mathbf{Rat}_S(S_0, \Sigma)$.

**Theorem** (Bloom-Ésik) *Kleene theorem for Conway semirings.* Let $S$ be a Conway semiring, $S_0$ a sub Conway semiring of $S$, and $\Sigma \subseteq S$. Then $\mathbf{Rat}_S(S_0, \Sigma) = \mathbf{Rec}_S(S_0, \Sigma)$.

The inclusion $\mathbf{Rec}_S(S_0, \Sigma) \subseteq \mathbf{Rat}_S(S_0, \Sigma)$ follows from the matrix star formula. The reverse inclusion is shown by establishing some closure properties of $\mathbf{Rec}_S(S_0, \Sigma)$.

## Kleene's theorem in Conway semirings

**Lemma** $\mathbf{Rec}_S(S_0, \Sigma)$ is closed under sum.

**Proof** Let $\mathbf{A} = (\alpha, A, \gamma)$ and $\mathbf{B} = (\beta, B, \delta)$ be automata over $(S_0, \Sigma)$. Define

$$\mathbf{A} + \mathbf{B} = \left( (\alpha, \beta), \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \right).$$

Then

$$|\mathbf{A} + \mathbf{B}| = (\alpha, \beta) \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \alpha A^* \gamma + \beta B^* \delta = |\mathbf{A}| + |\mathbf{B}|.$$

## Kleene's theorem in Conway semirings

**Lemma** $\mathbf{Rec}_S(S_0, \Sigma)$ is closed under product.

**Proof** Let $\mathbf{A} = (\alpha, A, \gamma)$ and $\mathbf{B} = (\beta, B, \delta)$ be automata over $(S_0, \Sigma)$. Define

$$\mathbf{A} \cdot \mathbf{B} = \left( (\alpha, 0), \begin{pmatrix} A & \gamma\beta B \\ 0 & B \end{pmatrix}, \begin{pmatrix} \gamma\beta\delta \\ \delta \end{pmatrix} \right).$$

Then

$$\begin{aligned} |\mathbf{A} \cdot \mathbf{B}| &= (\alpha, 0) \begin{pmatrix} A^* & A^*\gamma\beta B B^* \\ 0 & B^* \end{pmatrix} \begin{pmatrix} \gamma\beta\delta \\ \delta \end{pmatrix} \\ &= \alpha A^* \gamma\beta\delta + \alpha A^* \gamma\beta B^+ \delta \\ &= \alpha A^* \gamma\beta B^* \delta = |\mathbf{A}| \cdot |\mathbf{B}| \end{aligned}$$

## Kleene's theorem in Conway semirings

**Lemma** $\mathbf{Rec}_S(S_0, \Sigma)$ is closed under star.

**Proof** Let $\mathbf{A} = (\alpha, A, \gamma)$ be an automaton over $(S_0, \Sigma)$. Define

$$\mathbf{A}^* = \left( (\alpha, 1), \begin{pmatrix} (\gamma\alpha)^* A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} (\gamma\alpha)^*\gamma \\ 1 \end{pmatrix} \right).$$

Then

$$\begin{aligned} |\mathbf{A}^*| &= (\alpha, 1) \begin{pmatrix} ((\gamma\alpha)^* A)^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\gamma\alpha)^*\gamma \\ 1 \end{pmatrix} \\ &= \alpha((\gamma\alpha)^* A)^* (\gamma\alpha)^*\gamma + 1 \\ &= \alpha(\gamma\alpha + A)^*\gamma + 1 \\ &= \alpha(A^*\gamma\alpha)^* A^*\gamma + 1 \\ &= (\alpha A^*\gamma)^* \alpha A^*\gamma + 1 \\ &= (\alpha A^*\gamma)^* = |\mathbf{A}|^*. \end{aligned}$$

## Kleene's theorem in Conway semirings

**Proof of Kleene's thm, completed** By the above lemmas, the inclusion $\mathbf{Rat}_S(S_0, \Sigma) \subseteq \mathbf{Rec}_S(S_0, \Sigma)$ follows if each element of $S_0 \cup \Sigma$ is recognizable. But any $s \in S_0$ is the behavior of the automaton $(s, 0, 1)$. Also, any $a \in \Sigma$ is the behavior of

$$\left((1,0), \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

The proof is complete.

## Kleene's theorem in Conway semirings

Let $S$ be a Conway semiring and $A$ a set. Then $S\langle\langle A^*\rangle\rangle$ is a Conway semiring. Let $\Sigma$ denote the set $A$. We define $S^{\mathsf{rat}}\langle\langle A^*\rangle\rangle = \mathbf{Rat}_{S\langle\langle A^*\rangle\rangle}(S,\Sigma)$, $S^{\mathsf{rec}}\langle\langle A^*\rangle\rangle = \mathbf{Rec}_{S\langle\langle A^*\rangle\rangle}(S,\Sigma)$.

**Corollary** $S^{\mathsf{rat}}\langle\langle A^*\rangle\rangle = S^{\mathsf{rec}}\langle\langle A^*\rangle\rangle$.

When $S = \mathbb{B}$ and $A$ is an alphabet, this is Kleene's theorem.

## Partial Conway semirings

Sometimes the star operation is only partially defined, e.g., in the semirings $\mathbb{N}\langle\langle A^*\rangle\rangle$, where the star operation is only meaningful on the **proper series** (i.e., on those series mapping the empty word to 0.) The axiomatic may be extended to cover such semirings.

**Definition** (Bloom–Ésik–Kuich) A **partial *-semiring** is a semiring $S$ equipped with a partial star operation $^* : D(S) \to S$ whose domain of definition $D(S)$ is an **ideal** of $S$, i.e., $0 \in D(S)$, $D(S) + D(S) \subseteq D(S)$, $S \cdot D(S) \cdot S \subseteq D(S)$. A **partial Conway semiring** is a partial *-semiring which satisfies the sum star and product star identities:

$$
\begin{aligned}
(a+b)^* &= a^*(ba^*)^*, \quad a, b \in D(S)\\
(ab)^* &= 1 + a(ba)^*b, \quad a \text{ or } b \in D(S)
\end{aligned}
$$

## Complete semirings

**Definition** (Eilenberg) A **complete semiring** is a semiring $S$ equipped with a **summation operation** $\sum_{i\in I} s_i$ defined on all families $s_i,\ i \in I$ of elements of $S$ subject to the following axioms:

$$\sum_{i\in\emptyset} a_i = 0 \qquad \sum_{i\in\{1,2\}} a_i = a_1 + a_2$$

$$b(\sum_{i\in I} a_i) = \sum_{i\in I} ba_i, \qquad (\sum_{i\in I} a_i)b = \sum_{i\in I} a_i b$$

$$\sum_{j\in J} \sum_{i\in I_j} a_i = \sum_{i\in I} a_i$$

where in the last equation, $I$ is the disjoint union of the sets $I_j,\ j \in J$. A morphism of complete semirings is a semiring morphism which preserves all sums.

## Complete semirings

### Examples

1. The boolean semiring $\mathbb{B}$ with $\sum_{i \in I} s_i = 1$ iff $\exists i \ s_i = 1$.

2. The semiring $\mathbb{N}_\infty$ with $\sum_{i \in I} s_i = \infty$ iff $\exists i \ s_i = \infty$ or $\exists^\infty i \ s_i \neq 0$.

3. The lattice of all subsets of a set.

4. Any complete, completely distributive lattice.

## Complete semirings

**Definition** In any complete semiring $S$, we define a **star operation**: $s^* = \sum_{n \geq 0} s^n$, for all $s \in S$.

**Proposition** Any morphism of complete semirings preserves the star operation.

**Proposition** Any complete semiring is a Conway semiring.

## Complete semirings

**Proposition** Suppose that $S$ is a complete semiring.

• Then for each $n$, the matrix semiring $S^{n\times n}$, equipped with the pointwise summation is also a complete semiring. Moreover, the star operation determined by the complete semiring structure is the same as that determined by the matrix star formula.

• For each set $A$, the power series semiring $S\langle\langle A^*\rangle\rangle$, equipped with the pointwise summation is complete. Moreover, the star operation determined by the complete semiring structure agrees with the one defined earlier.

Thus, when $\mathbf{A} = (\alpha, A, \beta)$ over a complete semiring $S$, then $|\mathbf{A}| = \alpha M^*\beta = \alpha(\sum_{n\geq 0} M^n)\beta = \sum_{n\geq 0} \alpha M^n\beta$.

## Continuous semirings

A semiring $S$ is **ordered** if it is equipped with a partial order $\leq$ preserved by the operations of sum and product. A morphism of ordered semirings also preserves the partial order. A **positively ordered semiring** $S$ also satisfies $0 \leq a$ for all $a \in S$.

**Definition** (Eilenberg) A **continuous semiring** is a positively ordered semiring $S$ which is a cpo such that the sum and product operations are continuous. A morphism of continuous semirings is a morphism of ordered semirings which is a continuous function.

**Examples**

- Any finite positively ordered semiring.
- $\mathbb{N}_\infty$ and $\mathbb{B}$.
- The semiring of languages over a set $A$.
- Every complete, completely distributive lattice.

## Continuous semirings

**Proposition** Any continuous semiring is a complete semiring with summation

$$\sum_{i \in I} s_i = \bigvee_{F \subseteq I,\ F \text{ is finite}} \sum_{j \in F} s_j$$

Any morphism of continuous semirings is a complete semiring morphism.

Thus, there is a canonical star operation on each continuous semiring.

**Proposition** Any continuous semiring is a Conway semiring.

## Continuous semirings

**Proposition** Suppose that $S$ is a continuous semiring.

- Then for each $n$, the matrix semiring $S^{n \times n}$, equipped with the pointwise order is also a continuous semiring. Moreover, the star operation determined by the continuous semiring structure agrees with the one determined by the matrix star formula.
- For each set $A$, the power series semiring $S\langle\langle A^* \rangle\rangle$, equipped with the pointwise order is continuous. Moreover, the star operation determined by the continuous semiring structure agrees with the one defined earlier.

## Inductive $^*$-semirings

**Definition** (Ésik-Kuich) An **inductive $^*$-semiring** is an *ordered* semiring $S$ which is a $^*$-semiring such that for any $a, b \in S$, $a^*b$ is the least pre-fixed point of the function $S \to S$, $x \mapsto ax + b$:

- $aa^*b + b \leq b$ (or $aa^* + 1 \leq 1$)
- $ax + b \leq x \quad \Rightarrow \quad a^*b \leq x$

A **symmetric inductive $^*$-semiring** is an inductive $^*$-semiring whose *dual* is also an inductive $^*$-semiring. A morphism of (symmetric) inductive $^*$-semirings is an ordered semiring morphism which preserves star.

## Inductive *-semirings

**Proposition** Every continuous *-semiring is a symmetric inductive *-semiring. Every inductive *-semiring is a Conway semiring. Moreover, the star operation determined by the inductive semiring structure is the same as that determined by the matrix star formula.

**Proposition** If $S$ is a (symmetric) inductive *-semiring then, equipped with the pointwise order and the star operation defined above, so is each semiring $S^{n\times n}$ and $S\langle\langle A^*\rangle\rangle$. Moreover, the star operation determined by the inductive semiring structure agrees with the one defined earlier.

Thus, when $\mathbf{A} = (\alpha, A, \beta)$ is an automaton of dim. $n$ in an iductive semiring, $|\mathbf{A}| = \alpha A^* \beta$ with $A^*$ being the least solution of the matrix equation $X = AX + E_n$.

## Extensions and other results

**Extensions**: Automata on infinite words (Büchi automata), finite and infinite trees, algebraic theories.

**Completeness** results for the equational theory of (regular) languages, (rational) power series, tree languages and formal series of trees, and others.

## Some literature

S.L. Bloom, ZE: Iteration Theories, Springer-Verlag, 1993.

S.L. Bloom, ZE: Equational axioms for regular sets, *Mathematical Structures in Computer Science*, 3(1993), 1–24.

S.L. Bloom, ZE: Axiomatizing rational power series over natural numbers, *Information and Computation*, 207(2009), 793–811.

ZE, W. Kuich: Equational axioms for a theory of automata, in: *Formal Languages and Applications*, Springer-Verlag, 2004, 183–196.

ZE, W . Kuich: Free iterative and iteration $K$-semialgebras, *Algebra Universalis*, 67(2012), 141–162.

ZE, W. Kuich: Free inductive $K$-semialgebras, *J.of Logic and Algebraic Programming*, 82(2013), 111–122.