Packing up to 200 Equal Circles in a Square

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1 Introduction

The Hungarian mathematician Farkas Bolyai (1775–1856) published in his principal work ('Tentamen', 1832–33 [Bol04]) a dense regular packing of equal circles in an equilateral triangle (see Fig. 1). He defined an infinite packing series and investigated the limit of *vacuitas* (in Latin, the gap in the triangle outside the circles). It is interesting that these packings are not always optimal in spite of the fact that they are based on hexagonal grid packings. Bolyai probably was the first author in the mathematical literature who studied the density of a series of packing circles in a bounded shape.

The problem of finding the densest packing of n equal and non-overlapping circles has been studied for several shapes of the bounding region (e.g. in a rectangle, triangle and circle [Mel97]). This chapter focuses only on the 'Packing of Equal Circles in a Square'-problem (PECS problem), however, the developed stochastic optimization algorithm can be used for other shapes as well.

The Hungarian mathematicians Dezső Lázár and László Fejes Tóth have already investigated the PECS problem before 1940. This problem first appeared in the literature in 1960, when Leo Moser [Mos60] guessed the optimal arrangement of 8 circles. J. Schaer and A. Meir [SM65] proved this conjecture and J. Schaer [Sch65] solved the $n = 9$ case, too. C. de Groot et al. [GPW90] solved the $n = 10$ case after many authors published new and improved packings. R. Peikert et al. [PWMG92] found and proved optimal packings from $n = 10$ to $n = 20$ using a computer aided method. Based on theoretical tools only, G. Wengerodt published proofs for $n = 14, 16$, and 25 [Wen83, Wen87, Weng87], and with K. Kirchner for $n = 36$ [KW87]. However, there are gaps in both of proofs for $n = 25$ and 36 according to the review MR1453444 in Mathematical Reviews. In the last decades, several deterministic [NO99, LR02, Mar03, Mar04, MC05] and stochastic [NO97, BDGL00, CGSC01] methods were published for this problem.

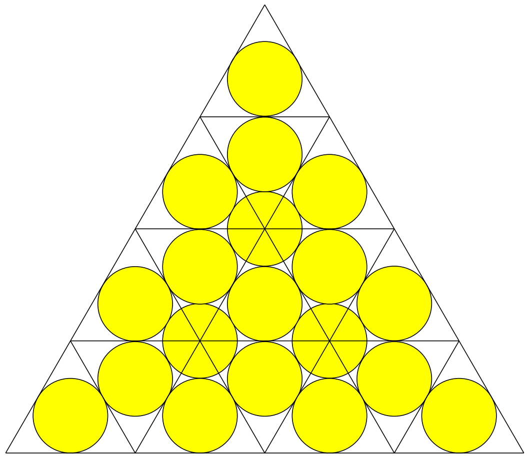

Fig. 1. The example of Bolyai for packing 19 equal circles in an equilateral triangle.

Proven optimal packings are known up to $n = 30$ [PWMG92, NO99, Mar03, Mar04, MC05]. Approximate packings (i.e. packings determined by computer aided numerical computations without a rigorous proof) were reported in the literature for up to $n = 100$ [NO97, BDGL00, CGSC01, LR02]. At the same time, some other related results (e.g. patterns, bounds and some properties of the optimal solutions) were published as well [GL96, NOS99, Sza00, SCCG01]. A more detailed history of the PECS problem can be found in [PWMG92, Mel97, SCCG01, SMC05].

In this chapter we propose many new approximate packings. These packings are interesting for discrete geometric investigations, because they suggest new possible structures. Note that similar structures occur in several cases, so they define pattern classes [GL96, NO97, NOS99, Sza00]. The good approximate packings are important for the reliable computer aided methods to speed up the localization of optimal packings and to prove the optimality [Mar03, NO99, PWMG92].

The chapter is organized as follows: Section 2 presents definitions and mathematical models of the PECS problem as a global optimization problem. In Section 3, a theoretical lower bound is given for every $n \geq 2$. We summarize some earlier deterministic and stochastic optimization approaches in Section 4. In the following Section 5, we propose a modified billiard simulation method and in Section 6 numerical values, figures and density plots of all optimal and approximate packings up to $n = 200$ are given. It has been verified by interval arithmetic based computations that the numerical results represent in fact existing packings.

2 Definitions and Models

First of all, we give a formal definition of the PECS problem.

Definition 2.1 Let us denote the number of the circles by $n \geq 2$. A *packing of circles with radius* r_n *in the unit square* is $P = (p_1, \ldots, p_n) \in P_{r_n}$, where $P_{r_n} = \{((x_1, y_1), \ldots, (x_n, y_n)) \in [0, 1]^{2n} \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4r_n^2; x_i, y_i \in [r_n, 1 - r_n] \ (1 \leq i < j \leq n)\}$. P is an *optimal packing of circles*, if $P \in P_{\overline{r}_n}$, where $\overline{r}_n = \max_{P_{r_n} \neq \emptyset} r_n$.

Consequently, this definition leads to the following

Problem (P): Determine the optimal packing of circles for $n \geq 2$ in the unit square.

From another point of view, we may consider only the centers of the circles, so that we obtain the following problem: Locate n points in the unit square in such a way that the minimal distance between any two of them be maximal.

Definition 2.2 Let us denote the number of the points by $n \geq 2$. A *point arrangement with a minimal distance* m_n *in the unit square* is $A = (a_1, \ldots, a_n) \in A_{m_n}$, where $A_{m_n} = \{((x_1, y_1), \ldots, (x_n, y_n)) \in [0, 1]^{2n} \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq m_n^2 \ (1 \leq i < j \leq n)\}$. A is an *optimal point arrangement*, if $A \in A_{\overline{m}_n}$, where $\overline{m}_n = \max_{A_{m_n} \neq \emptyset} m_n$.

This definition leads to

Problem (A): Determine the optimal point arrangements in the unit square for $n \geq 2$.

Problems (P) and (A) are known to be equivalent [Sza00]. The following relation holds between the radius $\overline{r}_n$ of the optimal packing and the distance $\overline{m}_n$ of the optimal point arrangement:

$$\overline{r}_n = \frac{\overline{m}_n}{2(\overline{m}_n + 1)}. \tag{1}$$

The following problem settings are also equivalent with problems (P) and (A), respectively:

- Find the smallest square of side $\overline{\rho}_n$ that contains n equal and non-overlapping circles with a radius of 1.
- Determine the smallest square of side $\overline{\sigma}_n$ that contains n points with mutual distances of at least 1.

Furthermore, it can be proved that

$$\overline{\rho}_n \overline{r}_n = 1 \qquad \text{and} \qquad \overline{\sigma}_n \overline{m}_n = 1. \tag{2}$$

The following definition summarizes some terms used in this chapter.

Definition 2.3 We say, that

- two circles are in *contact* in a packing if the distance between their centers is $2r_n$,
- a circle is *free* (a *rattler*) if it can be moved inside the square by a positive distance without getting in contact or overlapping another one,
- a circle is *fixed* if it isn't a free circle,
- the *density* of a packing in the unit square is $d_n = nr_n^2\pi$.

2.1 The PECS Problem as a Global Optimization Problem

The PECS problem is on the one hand a geometrical problem and on the other hand a continuous global optimization problem [TZ89]. Problem (A) can be written shortly as a $2n$ dimensional optimization problem in the following form:

$$\max_{s_k \in [0,1]^2,\ 1 \le k \le n} \ \min_{1 \le i < j \le n} \|s_i - s_j\|.$$

This problem can be considered as

a) *a continuous nonlinear constrained global optimization problem*:

$$\max_{x_i, y_i} \ t$$

subject to

$$\sqrt{(x_i - x_j)^2 - (y_i - y_j)^2} \ge t \qquad (1 \le i < j \le n)$$

$$0 \le x_i, y_i \le 1, \qquad 1 \le i \le n.$$

b) *a max-min optimization problem*:

$$\max_{x_i, y_i} \ \min_{1 \le i < j \le n} s_{ij}$$

subject to

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} = s_{ij} \qquad (1 \le i < j \le n)$$

$$0 \le x_i, y_i \le 1, \qquad 1 \le i \le n.$$

c) *a DC programming problem* [6]:

A DC (difference of convex functions) programming problem is a mathematical programming problem, where the objective function can be given as a difference of two convex functions. The objective function of the PECS problem can be stated as the difference of the convex functions g and h:

$$g(z) = 2\sum_{j=1}^{2n} z_j^2, \quad \text{and}$$

$$h(z) = \max\left\{\left(2\sum_{j\in J\setminus J_{ik}} z_j^2 + (z_i+z_k)^2 + (z_{n+i}+z_{n+k})^2\right) \,:\, 1 \le i < k \le n\right\},$$

where $J = \{1,\dots,2n\}$, $z = (x_1,\dots,x_n,y_1,\dots,y_n)$, $J_{ik} = \{i,k,n+i,n+k\}$.

d) and finally as *an all-quadratic optimization problem.*

The general form of an all-quadratic optimization problem is

$$\min \quad \left[x^{\mathrm{T}} Q^0 x + (d^{\,0})^{\mathrm{T}} x\right]$$

$$\text{subject to}$$

$$x^{\mathrm{T}} Q^{\,l} x + (d^{\,l})^{\mathrm{T}} x + c^{\,l} \le 0 \qquad l = 1,\dots,p$$

$$x \in P,$$

where $Q^{\,l}$ $(l = 0,\dots,p)$ are real $(n+1)\times(n+1)$ matrices, $d^{\,l}$ $(l = 0,\dots,p)$ are real $(n+1)$-dimensional vectors, $c^{\,l}$ $(l = 1,\dots,p)$ are real numbers, p is the number of constraints, and P is a polyhedron. Solving the general case of an all-quadratic optimization problem is knew to be NP-hard.

The PECS problem with the following values is a special all-quadratic optimization problem with a linear objective function:

$$Q^0 = \mathbf{0}, \quad x^{\mathrm{T}} = (x_0, x_1, \dots, x_{2n}), \quad (d^{\,0})^{\mathrm{T}} = (-1, 0, \dots, 0),$$

$$(d^{\,l})^{\mathrm{T}} = \underline{\mathbf{0}}, \;\; c^{\,l} = 0, \;\; p = \frac{n(n-1)}{2}, \;\; P = [0,\sqrt{2}] \times [0,1]^{2n},$$

and for all

$$1 \le i, j \le 2n+1,$$

$$1 \le l' < l'' \le n.$$

$$[Q^l]_{ij} = Q_{ij}^{l'l''} = \begin{cases} -1, & \text{if } i = j = \begin{cases} 2l', \\ 2l'', \\ 2l'+1, \\ 2l''+1, \end{cases} \\ 1, & \text{if } i = j = 1, \\ & i = 2l''+1 \text{ and } j = 2l'+1, \\ & i = 2l'' \quad \text{and } j = 2l', \\ & i = 2l'+1 \text{ and } j = 2l''+1, \\ & i = 2l' \quad \text{and } j = 2l'', \\ 0, & \text{otherwise.} \end{cases}$$

In this model, x_0 is the minimal distance between the points. The coordinates of the i^{th} point $(1 \le i \le n)$ are (x_{2i-1}, x_{2i}).

The investigations show that those approaches are effective that use not only optimization models, but some geometrical properties of the problem, too. Before we study the most efficient earlier approaches, we give some theoretical lower bounds for $\overline{m}_n$.

3 Theoretical Lower Bound for $\overline{m}_n$

In this section we show a proof for the statement that the known $\sqrt{\frac{2}{\sqrt{3}n}}$ asymptotic formula is a lower bound of $\overline{m}_n$.

Proposition 1. *For every integer $n \ge 2$*

$$\sqrt{\frac{2}{\sqrt{3}n}} < \overline{m}_n.$$

Proof: Proposition 1 is equivalent with the following statement:

There can be at least $\frac{2}{\sqrt{3}}\sigma^2$ points located in a square of side σ such that the distances between the pairs of points are at least 1.

The proof is constructive, and it is based on the hexagonal grid packing. Let us divide the square into stripes of width $\frac{\sqrt{3}}{2}$. This division determines $\left\lfloor \frac{2\sigma}{\sqrt{3}} \right\rfloor + 1$ parallel zones. The first level is one side of the square. On this level $\lfloor \sigma \rfloor + 1$ points can be located, where the first point is in the edge of the square and the distance between the points is 1. On the second level the distance between the side of the square and the first point is $\frac{1}{2}$. On this level one can place $\left\lfloor \sigma - \frac{1}{2} \right\rfloor + 1$ points. If the number of the level is an odd number, then there are always $\lfloor \sigma \rfloor + 1$ points on this level and for an even number there are $\left\lfloor \sigma - \frac{1}{2} \right\rfloor + 1$ points. If $\left\lfloor \frac{2\sigma}{\sqrt{3}} \right\rfloor + 1$ is even, then there are

$\frac{\left\lfloor \frac{2\sigma}{\sqrt{3}} \right\rfloor + 1}{2} \left(\lfloor \sigma \rfloor + \left\lfloor \sigma - \frac{1}{2} \right\rfloor + 2 \right)$ points in the square. If it is odd, then there are $\frac{\left\lfloor \frac{2\sigma}{\sqrt{3}} \right\rfloor}{2} \left(\left\lfloor \sigma - \frac{1}{2} \right\rfloor + \lfloor \sigma \rfloor + 2 \right) + \lfloor \sigma \rfloor + 1$ points. In both cases it is easy to see that the stated inequality holds, because $2\sigma < \left\lfloor \sigma - \frac{1}{2} \right\rfloor + \lfloor \sigma \rfloor + 2$, $\frac{2}{\sqrt{3}}\sigma - 1 < \left\lfloor \frac{2\sigma}{\sqrt{3}} \right\rfloor$ and have $\sigma - 1 < \lfloor \sigma \rfloor$, therefore

$$\frac{\frac{2\sigma}{\sqrt{3}} - 1 + 1}{2} 2\sigma < n \quad \Longrightarrow \quad \frac{2}{\sqrt{3}}\sigma^2 < n \quad \text{(even case)},$$

and

$$\frac{\frac{2\sigma}{\sqrt{3}} - 1}{2} 2\sigma + \sigma - 1 + 1 < n \quad \Longrightarrow \quad \frac{2}{\sqrt{3}}\sigma^2 < n \quad \text{(odd case)}.$$

□

The theoretical lower bounds of $\overline{m}_n$ were well improved by computer aided methods. These approaches usually fall into two classes: deterministic and stochastic methods. A typical way to find approximate packings in the latter case is to apply stochastic global optimization algorithms.

4 Some Earlier Approaches for Finding Approximate Packings

In this section we give an overview of some earlier methods to find approximate packings. Several strategies were used (e.g. nonlinear programming solvers, and the Cabri-Geométry software). Here we summarize some useful earlier approaches to find approximate packings for higher n values.

4.1 Energy Function Minimization

The PECS problem as a mathematical programming problem can be formalized in the following way:

$$\max \quad \min_{1 \le i < j \le n} \|s_i - s_j\|$$

subject to

$$s_i \in [0,1]^2, \;\; 1 \le i \le n.$$

By virtue of $\min_{1 \le i < j \le n} \|s_i - s_j\| = \lim_{m \to -\infty} \left(\sum_{1 \le i < j \le n} \|s_i - s_j\|^m \right)^{\frac{1}{m}}$, the problem is relaxed as

$$\min_{s_i \in [0,1]^2, \;\; 1 \le i \le n} \quad \sum_{1 \le i < j \le n} \frac{1}{\|s_i - s_j\|^m}.$$

This objective function can be interpreted as a potential or energy function. A physical analogon of this approach is to regard the points as electrical charges (all positive or all negative) which are repulsing eachother. If the minimal distance between the charged particles increases, the corresponding value of the potential function decreases. K. J. Nurmela and P. R. J. Östergård [NO97] used a similar energy function with large values of m:

$$\sum_{1\le i<j\le n} \left(\frac{\lambda}{||s_i - s_j||^2} \right)^m .$$

Introducing $x_i = \sin(x_i^{'})$ and $y_i = \sin(y_i^{'})$, it transforms into an unconstrained optimization problem in variables $x_i^{'}, y_i^{'}$, where the coordinates of the centers of the circles fulfill the constraints $-1 \le x_i \le 1$, $-1 \le y_i \le 1$. They published candidate packings up to 50 circles using a combination of the Goldstein-Armijo backtracking linear search and the Newton method for the optimization.

4.2 Billiard Simulation

Let us consider a random arrangement of the points. Draw equal circles around the points without overlapping. Each circle can be considered as a ball with an initial radius, moving direction, and speed. Start the balls and increase slowly the common radius of them. The swing of each ball during the process will be less and less. The algorithm stops when the packing or a substructure of the packing becomes rigid for even. Using billiard simulation, R. L. Graham and B. D. Lubachevsky [GL96] reported several approximate packings for up to 50 circles and for some values beyond.

4.3 A Perturbation Method

D. W. Boll et al. [BDGL00] used a stochastic algorithm which gave improved packings for $n = 32, 37, 48$, and 50. A brief outline of their method is

1. *Step:* consider n random points in the unit square and define $s = 0.25$,
2. *Step:* for each point
   - *a)* perturb the place of the center by s in North, South, East or West direction,
   - *b)* if during the movement the distance between the point and its nearest neighbour becomes greater, update the new location of the point,
3. *Step* repeat *Step 2* while movable points exist,
4. *Step* $s := s/1.5$, if $s > 10^{-10}$, and continue with *Step 2.*

Using the previous simple algorithm good candidate packings can be found after some millions of iterations. They have found unpublished approximate packings up to $n = 200$.

4.4 TAMSASS-PECS

The TAMSASS-PECS (Threshold Accepting Modified Single Agent Stochastic Search for Packing Equal Circles in a Square) method is based on the Threshold Accepting global optimization technique and a modified SASS local optimization algorithm. The algorithm starts with a pseudorandom initial packing with certain set, standard deviation and threshold level. The algorithm tries to improve the current solution by an iterative procedure. At every step it tries to find a better position of the actual point using a local search. The stopping criterion is based on the value of the standard deviation, which is decreased at every iteration. The framework of the method is the Threshold Accepting approach. It is a close alternative of the Simulated Annealing algorithms. It accepts every move that leads to a new approximate solution not much worse than the current one and rejects other moves. Using TAMSASS-PECS L. G. Casado et al. [CGSC01] reported approximate packings up to $n = 100$ and improved some earlier packings.

4.5 A Deterministic Approach Based on LP-relaxation

The PECS problem can be regarded as an all-quadratic optimization problem, i.e. an optimization problem with not necessarily convex quadratic constraints. The hardness is due to the large number of constraints. This approach provides a rectangular subdivision branch-and-bound algorithm. To provide an upper bound at each node of the branch-and-bound tree, M. Locatelli and U. Raber used the special structure of the constraints and gave an LP-relaxation [LR02]. They have found candidate packings up to 39 circles proving the optimality theoretically within a given accuracy, except when $n = 36$ and 37. Moreover, a new solution for $n = 37$ has been detected, but not yet proved to be optimal within the given tolerance.

5 The MBS Algorithm

We will call our approach MBS (Modified Billiard Simulation). The basic idea is as follows: Distribute randomly n points inside the unit square and blow them up as increasing circles in a uniform manner. This can be done by incrementing the radii gradually from an initial value, which is a safe lower bound. In early stages of the process, when the distance between the small circles is much greater than their size and no collisions occur, there is no need to change their positions. As the circles grow, we have to deal with collisions between them, and with those between the circles and the boundaries. The iteration stops when the improvement is too small or the number of iterations is larger than a given number.

The efficiency of the MBS algorithm comes from a significant reduction of computational costs. The basic idea is as follows: It is not necessary to calculate and store the distance between two circles if they are too far from each other and will never meet. For the numerical calculation the program uses two matrices CCD and CED. Matrix CCD stores the adjacency between the objects themselves, and the matrix CED holds this between the objects and the sides of the square. At start, all matrix elements are set to NEAR which implies that only such pairs of circles will be checked during the calculation. When after thousands of collosions a mutual distance of a pair is great enough, then the value is set to FAR which means that this contact will never occur in later iterations. As long as the program runs the cost of the subroutine which determines the contacts will become less and less.

It is useful to consider not only random arrangements for the initial packing but hexagonal or other regular lattice packings too. Sometimes the relationship between the number of the circles and the structure of packing can predict a good initial configuration. The code and the found packings can be downloaded from the Packomania web-site: `http://www.packomania.com/`.

6 Approximate Packings up to n=200

In this section the found packings are reported up to $n = 200$ using the MBS algorithm. The packings were found by numerical computations. An important part of the investigations is to prvoide a guaranty that the arrangements with the given structures really (in mathematical sense) exist. A possible way for proving is based on a system of equations which corresponds to the packings. The solution of the system of equations (if it exists) prove rigorously the existence of the packing. To solve the system of equations is easy if we can guess the solution. Sometimes the structures of the packings help us to guess the exact coordinates of the circles (e.g. for grid packings). For the present situation any kind of reliable tools can be used (e.g. the software package INTBIS) to solve the system of equations.

The numerical value of m_n could be incorrect due to rounding, so we checked the calculations by interval arithmetic computations, too. Based on these computations the new m_n value is the minimum of the lower bounds of the mutual interval distances between the points. In a formula it is

$$m_n = \min_{1 \le i < j \le n} \underline{D_{ij}},$$

where

$$D_{ij} = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2},$$

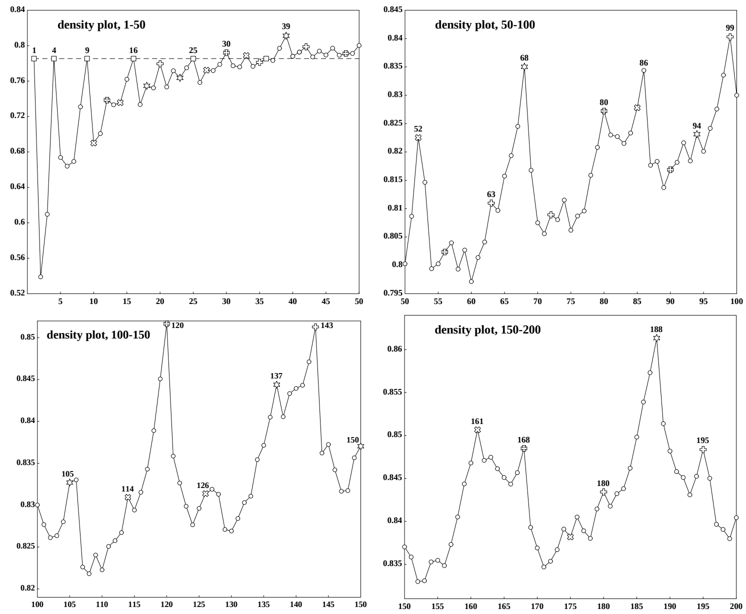

Fig. 2. Density plots for up to 200 circles.

$$X_i = [x_i - \epsilon, x_i + \epsilon], \quad Y_i = [y_i - \epsilon, y_i + \epsilon],$$
$$X_j = [x_j - \epsilon, x_j + \epsilon], \quad Y_j = [y_j - \epsilon, y_j + \epsilon].$$

and (x_i, y_i) is the coordinates of the ith point $(1 \leq i \leq n)$. Underlining means the lower bound. The capital letters denote intervals. The accuracy of the calculations was 10^{-14}. A fully interval arithmetic based method to validate the solutions was proposed by M.Cs. Markót [Mar03, Mar04, MC05].

Fig. 2 shows the density plots of packings for up to 200 circles. It is interesting to observe the local maxima of the graphs. Usually the related packings are grid or near-grid packings.

Tables 1–3 use bold face for the improved packings compared to other, previous results, and italic style denotes that there are better known values, but the coordinates have not been published yet. In 66 cases we have found similar values to the packings reported in the literature. In 24 cases the results are worse than Dave Boll et al.'s packings. The number of the improved lower bounds of the packings is 110. In these tables we summarized the numerical results.

Table 1. The numerical results for $n = 2 - 36$ circles.

n	exact r_n	exact m_n	approx. m_n	d_n
2	$\frac{1}{2}(2-\sqrt{2})$	$\sqrt{2}$	1.4142135624	0.5390120845
3	$\frac{1}{2}(8-5\sqrt{2}+4\sqrt{3}-3\sqrt{6})$	$\sqrt{6}-\sqrt{2}$	1.0352761804	0.6096448087
4	$\frac{1}{4}$	1	1.0000000000	0.7853981634
5	$\frac{1}{2}(-1+\sqrt{2})$	$\frac{1}{2}m_2$	0.7071067812	0.6737651056
6	$\frac{1}{46}(-13+6\sqrt{13})$	$\frac{1}{6}\sqrt{13}$	0.6009252126	0.6639569095
7	$\frac{1}{13}(4-\sqrt{3})$	$4-2\sqrt{3}$	0.5358983849	0.6693108268
8	$\frac{1}{4}(1+\sqrt{2}-\sqrt{3})$	$\frac{1}{2}m_3$	0.5176380902	0.7309638253
9	$\frac{1}{6}$	$\frac{1}{2}$	0.5000000000	0.7853981634
10	–	–	0.4212795440	0.6900357853
11	(see below)	(see below)	0.3982073102	0.7007415778
12	$\frac{1}{382}(-34+15\sqrt{34})$	$\frac{1}{15}\sqrt{34}$	0.3887301263	0.7384682239
13	–	–	0.3660960077	0.7332646949
14	$\frac{1}{33}(6-\sqrt{3})$	$\frac{2}{13}(4-\sqrt{3})$	0.3489152604	0.7356792555
15	$\frac{1}{2}r_3$	$2\,r_8$	0.3410813774	0.7620560109
16	$\frac{1}{8}$	$\frac{1}{3}$	0.3333333333	0.7853981634
17	–	–	0.3061539853	0.7335502633
18	$\frac{1}{262}(-13+12\sqrt{13})$	$\frac{1}{2}m_6$	0.3004626063	0.7546533579
19	–	–	0.2895419920	0.7523078967
20	$\frac{1}{482}(65-8\sqrt{2})$	$\frac{1}{16}(6-\sqrt{2})$	0.2866116524	0.7794936869
21	–	–	0.2718122554	0.7533577029
22	–	–	0.2679584016	0.7716801121
23	$\frac{1}{2}(-7-5\sqrt{2}+4\sqrt{3}+3\sqrt{6})$	$\frac{1}{4}m_3$	0.2588190451	0.7636310321
24	$\frac{1}{92}(21-5\sqrt{2}+3\sqrt{3}-4\sqrt{6})$	r_3	0.2543330950	0.7749632598
25	$\frac{1}{10}$	$\frac{1}{4}$	0.2500000000	0.7853981634
26	–	–	0.2387347572	0.7584690905
27	$\frac{1}{3022}(-89+40\sqrt{89})$	$\frac{1}{40}\sqrt{89}$	0.2358495283	0.7723114565
28	–	–	0.2305354936	0.7718541114
29	–	–	0.2268829007	0.7789062418
30	$\frac{1}{1202}(126-5\sqrt{10})$	$\frac{1}{75}(20-\sqrt{10})$	0.2245029645	0.7920190265
31	–	–	0.2175472916	0.7772974787
32	–	–	0.2131341934	0.7757618736
33	–	–	0.2113283841	0.7888523039
34	$\frac{1}{2}m_8$	$2r_{23}$	0.2056046467	0.7766490643
35	(see below)	$2\,r_{24}$	0.2027636009	0.7812272130
36	$\frac{1}{12}$	$\frac{1}{5}$	0.2000000000	0.7853981634

$$r_{11} = \frac{1}{568}\left(176 - 9\sqrt{2} - 14\sqrt{3} - 13\sqrt{6} - 2\sqrt{-16523 + 12545\sqrt{2} - 9919\sqrt{3} + 6587\sqrt{6}}\right)$$

$$m_{11} = \frac{1}{4}\left(-4 - 3\sqrt{2} + 2\sqrt{3} + 3\sqrt{6} + \left(4 + \sqrt{2} - 2\sqrt{3} - \sqrt{6}\right)\sqrt{1 + 2\sqrt{2}}\right)$$

$$r_{35} = \frac{1}{772}(112 - 17\sqrt{2} + 8\sqrt{3} - 15\sqrt{6})$$

Table 2. The numerical results for $n = 37 - 124$ circles.

n	approx. m_n	d_n	n	approx. m_n	d_n
37	0.1964291842	0.7833027206	**81**	0.1283368559	0.8230005836
38	0.1953423041	0.7970425568	**82**	0.1274269118	0.8227146966
39	0.1943650631	0.8111790272	**83**	0.1264543531	0.8215014743
40	0.1881755220	0.7879795188	**84**	0.1257627022	0.8233399269
41	0.1860995118	0.7927238993	85	0.1253084290	0.8278015474
42	0.1842770721	0.7986842786	*86*	0.1250425340	0.8343840262
43	0.1801911354	0.7872641664	**87**	0.1228265796	0.8176519577
44	0.1786392456	0.7938428078	**88**	0.1220983545	0.8183334905
45	0.1757163141	0.7894442684	**89**	0.1209431296	0.8137202694
46	0.1744593608	0.7971871319	90	0.1204480494	0.8168616061
47	0.1712705638	0.7892938820	**91**	0.1198126022	0.8181738104
48	0.1694054293	0.7911440338	**92**	0.1193622666	0.8216190446
49	0.1673860768	0.7912169895	**93**	0.1183841706	0.8184234765
50	0.1665265773	0.8002721839	**94**	0.1180563847	0.8231316147
51	0.1656183743	0.8086569481	**95**	0.1171194312	0.8201101448
52	0.1653862379	0.8225308420	96	0.1167579991	0.8241689679
53	0.1626480663	0.8146424042	**97**	0.1163574839	0.8276442131
54	0.1591395163	0.7994076230	**98**	0.1165351364	0.8335521941
55	0.1575557475	0.8002712972	**99**	0.1160181348	0.8402999370
56	0.1561565004	0.8023517295	**100**	0.1145801945	0.8300157794
57	0.1547474069	0.8039656738	**101**	0.1137678096	0.8276766917
58	0.1526925313	0.7993307243	**102**	0.1130265581	0.8261170420
59	0.1515619183	0.8026893318	**103**	0.1124324277	0.8263480419
60	0.1495056540	0.7971391564	**104**	0.1119552283	0.8280134249
61	0.1485441266	0.8013741245	**105**	0.1117113035	0.8327015567
62	0.1474526798	0.8041134778	**106**	0.1111479979	0.8330195984
63	0.1468193136	0, 8109737806	*107*	0.1098006344	0.8226089313
64	0.1453677544	0.8096850385	108	0.1091766948	0.8218111889
65	0.1446990147	0.8157400181	109	0.1087836926	0.8240438322
66	0.1438039660	0.8193554530	110	0.1081056069	0.8222742690
67	0.1430855758	0.8245156558	**111**	0.1077672183	0.8250669455
68	0.1429094775	0.8350205982	**112**	0.1072831265	0.8257591285
69	0.1399481818	0.8167765737	*113*	0.1068253164	0.8267201679
70	0.1379067766	0.8075060206	**114**	0.1066053963	0.8309359404
71	0.1366129972	0.8055769545	**115**	0.1059840023	0.8294126663
72	0.1358499279	0.8089083021	**116**	0.1056270416	0.8315355075
73	0.1347098276	0.8080563922	**117**	0.1053200675	0.8342993237
74	0.1339986726	0.8115167751	**118**	0.1051451721	0.8389032937
75	0.1324888136	0.8061980168	**119**	0.1050811970	0.8450812908
76	0.1317300376	0.8086999566	**120**	0.1050454468	0.8516581650
77	0.1308410780	0.8095910198	**121**	0.1034896517	0.8358581333
78	0.1304607726	0.8158933303	*122*	0.1028022705	0.8328448925
79	0.1299652027	0.8208069227	**123**	0.1021525019	0.8298691626
80	0.1296133854	0.8272178885	**124**	0.1015480773	0.8276525614

Table 3. The numerical results for $n = 125 - 200$ circles.

n	approx. m_n	d_n	n	approx. m_n	d_n
125	0.1012318916	0.8296158463	**163**	0.0885685524	0.8474713225
126	0.1009062359	0.8313727312	**164**	0.0881985853	0.8461369376
127	0.1005031141	0.8318980160	**165**	0.0878493719	0.8451107320
128	0.1000307339	0.8312987414	**166**	0.0875180593	0.8443458471
129	0.0993263118	0.8270941821	**167**	0.0873073510	0.8456746787
130	0.0988937215	0.8269119921	**168**	0.0871824504	0.8485011578
131	0.0986288136	0.8292143817	**169**	0.0863882436	0.8392971739
132	0.0982888835	0.8303084177	**170**	0.0859789843	0.8369135174
133	0.0979321354	0.8310764541	*171*	0.0855822297	0.8346948524
134	0.0978107863	0.8354360146	*172*	0.0853489718	0.8353646965
135	0.0975218520	0.8371459461	**173**	0.0851552703	0.8367106190
136	0.0973414248	0.8405056543	**174**	0.0850212754	0.8391079841
137	0.0971962814	0.8443861367	**175**	0.0847072864	0.8381936262
138	0.0965684977	0.8405593584	**176**	0.0845723809	0.8405094143
139	0.0963607140	0.8433304280	**177**	0.0842258678	0.8389085248
140	0.0960210816	0.8439433382	**178**	0.0839212683	0.8380279161
141	0.0956706024	0.8443178921	**179**	0.0838518837	0.8414507200
142	0.0954741493	0.8471212060	**180**	0.0837056682	0.8434307533
143	0.0953634647	0.8512820167	**181**	0.0833661059	0.8417768920
144	0.0940769937	0.8362256095	*182*	0.0831953206	0.8432289693
145	0.0937839766	0.8372440265	*183*	0.0829799252	0.8438130489
146	0.0932472250	0.8342146320	*184*	0.0828621097	0.8462006652
147	0.0927445712	0.8316620732	*185*	0.0828104531	0.8498202051
148	0.0924063478	0.8317384492	**186**	0.0827835097	0.8539004243
149	0.0923037281	0.8356565148	**187**	0.0827219524	0.8573124997
150	0.0920498844	0.8370331951	**188**	0.0826931189	0.8613421953
151	0.0916459689	0.8358530518	**189**	0.0819385675	0.8513795535
152	0.0911461597	0.8329987760	**190**	0.0815394198	0.8481916850
153	0.0908262169	0.8330913575	**191**	0.0811842378	0.8457992069
154	0.0906340269	0.8352857436	**192**	0.0809200493	0.8451158427
155	0.0903257963	0.8354733542	*193*	0.0805888840	0.8430949394
156	0.0899733630	0.8348541653	*194*	0.0804752001	0.8452518615
157	0.0898047212	0.8373181432	**195**	0.0804115030	0.8483644354
158	0.0896806181	0.8405154516	**196**	0.0800186870	0.8450185742
159	0.0895956726	0.8443652173	*197*	0.0795255746	0.8396607811
160	0.0894304663	0.8468018971	*198*	0.0792782634	0.8390666657
161	0.0893487418	0.8506653983	*199*	0.0790085605	0.8379950934
162	0.0888450720	0.8471092098	**200**	0.0789188489	0.8404343621

The figures of the packings are available in PostScript files including the coordinates of the circles at the mentioned web-site. Fig. 3 represent two nice approximate packings with free circles.

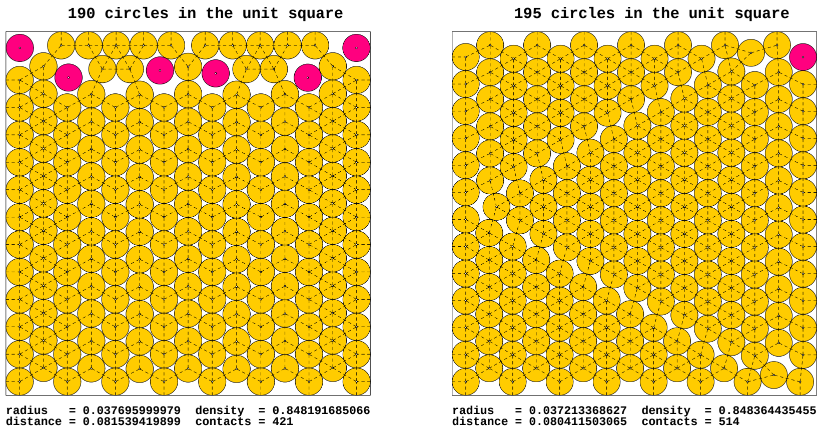

Fig. 3. Approximate packings for 190 and 195 circles.

Summary

The chapter gives a short overview of some earlier models and computer aided methods for the packing of equal circles in a square problem. Theoretical lower bounds are given for every n and based on a modified billiard simulation algorithm several approximate packings are reported up to $n = 200$.

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