INTERVAL ANALYSIS: ALGORITHMIC IMPROVEMENTS USING A HEURISTIC PARAMETER, REJECTINDEX FOR INTERVAL OPTIMIZATION

Interval optimization methods (**Interval analysis: Unconstrained and Constrained Optimization**) have the guarantee not to loose global optimizer points. To achieve this, a deterministic branch-and-bound framework is applied. Still heuristic algorithmic improvements may increase the convergence speed while keeping the guaranteed reliability.

The indicator parameter called *RejectIndex*

$$pf^*(X) = \frac{f^* - \underline{F}(X)}{\overline{F}(X) - \underline{F}(X)}.$$

was suggested by `L. G. Casado` as a measure of the closeness of the interval X to a global minimizer point [1]. First it was applied to improve the work load balance of global optimization algorithms.

A subinterval X of the search space with the minimal value of the inclusion function $\underline{F}(X)$ is usually considered as the best candidate to contain a global minimum. However, the larger the interval X, the larger the overestimation of the range $f(X)$ on X compared to $F(X)$. Therefore a box could be considered as a good candidate to contain a global minimum just because it is larger than the others. In order to compare subintervals of different size we normalize the distance between the global minimum value f^* and $\underline{F}(X)$.

The idea behind pf^* is that in general we expect the overestimation to be symmetric, i.e., the overestimation above $f(X)$ is closely equal to the overestimation below $f(X)$ for small subintervals containing a global minimizer point. Hence, for such intervals X the relative place of the global optimum value inside the $F(X)$ interval should be high, while for intervals far from global minimizer points pf^* must be small. Obviously, there are exceptions, and there exists no theoretical proof that pf^* would be a reliable indicator of nearby global minimizer points.

The value of the global minimum is not available in most cases. A generalized expression for a wider class of indicators is:

$$p(\hat{f}, X) = \frac{\hat{f} - \underline{F}(X)}{\overline{F}(X) - \underline{F}(X)},$$

where the $\hat{f}$ value is a kind of approximation of the global minimum. We assume that $\hat{f} \in F(X)$, i.e., this estimation is realistic in the sense that $\hat{f}$ is within the known bounds of the objective function on the search region. According to the numerical experience collected, we need a good approximation of the f^* value to improve the efficiency of the algorithm.

Subinterval selection.

I. Among the possible applications of these indicators the most promising and straightforward is in the *subinterval selection*. The theoretical and computational properties of the interval branch-and-bound optimization has been investigated extensively [6, 7, 8, 9]. The most important statements proved are the following for algorithms with balanced subdivision direction selection:

1. Assume that the inclusion function of the objective function is isotone, it has the zero convergence property, and the $p(f_k, Y)$ parameters are calculated with the f_k parameters converging to $\hat{f} > f^*$, for which there exists a point $\hat{x} \in X$ with $f(\hat{x}) = \hat{f}$. Then the branch-and-bound algorithm that selects that interval Y from the working list which has the maximal $p(f_i, Z)$ value can converge to a point $\hat{x} \in X$ for which $f(\hat{x}) > f^*$, i.e., to a point which is not a global minimizer point of the given problem.

2. Assume that the inclusion function of the objective function has the zero convergence property, and f_k converges to $\hat{f} < f^*$. Then the optimization branch-and-bound algorithm will produce an everywhere dense sequence of subintervals converging to each point of the search

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region X regardless of the objective function value.

3. Assume that the inclusion function of the objective function is isotone and has the zero convergence property. Consider the interval branch-and-bound optimization algorithm that uses the *cut-off test*, the *monotonicity test*, the *interval Newton step* and the *concavity test* as accelerating devices, and that selects as next leading interval that interval Y from the working list which has the maximal $p(f_i, Z)$ value. A necessary and sufficient condition for the convergence of this algorithm to a set of global minimizer points is that the sequence $\{f_i\}$ converges to the global minimum value f^*, and there exist at most a finite number of f_i values below f^*.

4. If our algorithm applies the interval selection rule of maximizing the $p(f^*, X) = pf^*(X)$ values for the members of the list L (i.e., if we can use the known exact global minimum value), then the algorithm converges exclusively to global minimizer points.

5. If our algorithm applies the interval selection rule of maximizing the $p(\tilde{f}, X)$ values for the members of the list L, where $\tilde{f}$ is the best available upper bound for the global minimum, *and its convergence to f^* can be ensured*, then the algorithm converges exclusively to global minimizer points.

6. Assume that for an optimization problem $\min_{x \in X} f(x)$ the inclusion function $F(X)$ of $f(x)$ is isotone and α-convergent with given positive constants α and C. Assume further that the pf^* parameter is less than 1 for all the subintervals of X. Then an arbitrary large number $N(> 0)$ of consecutive leading intervals of the basic B&B algorithm that selects the subinterval with the smallest lower bound as next leading interval may have the properties that:

i, none of these processed intervals contains a stationary point, and

ii, during this phase of the search the pf^* values are maximal for these intervals.

7. Assume that the inclusion function of the objective function is isotone and it has the zero convergence property. Consider the interval branch-and-bound optimization algorithm that uses the cut-off test, the monotonicity test, the interval Newton step and the concavity test as accelerating devices, and that selects as next leading interval that interval Y from the working list which has the maximal $pf(f_k, Z)$ value.

i, The algorithm converges exclusively to global minimizer points if

$$\underline{f}_k \le f_k < \delta(\overline{f}_k - \underline{f}_k) + \underline{f}_k$$

holds for each iteration number k, where $0 < \delta < 1$.

ii, The above condition is sharp in the sense that $\delta = 1$ allows convergence to not optimal points.

Here $\underline{f}_k = \min\{\underline{F}(Y^l), l = 1, ..., |L_k|\} \le f_k < \tilde{f}_k = \overline{f}_k$, where $|L|$ stands for the cardinality of the elements of the list L.

II. These theoretical results are in part promising (e.g. 7.), in part disappointing as 5. and 6. The conclusions of the detailed numerical comparisons were that if the global minimum value is known then the use of the pf^* parameter in the described way can accelerate the interval optimization method by orders of magnitude, and this improvement is especially strong for hard problems.

In case the global minimum value is not available, then its estimation, f_k that fulfills the conditions of 7. can be utilized with similar efficacy, and again the best results were achieved on difficult problems.

Multisection. I. The technique *multisection* is a way to accelerate branch-and-bound methods by subdividing the actual interval into several subintervals in a single algorithm step. In the extreme case half of the function evaluations can be saved [5, 10]. On the basis of the RejectIndex value of a given interval it is decided, whether

cut-off test
monotonicity test
interval Newton step
concavity test
multisection

simple bisection or two higher degree multisection is to be applied [2, 11]. Two threshold values, $0 < P_1 < P_2 < 1$ are used for selecting the proper multisection type.

This algorithm improvement can also be cheated in the sense that there exist global optimization problems for which the new method will follow for an arbitrary long number of iteration an embedded interval sequence that contains no global minimizer point, or that intervals in which there is a global minimizer have misleading indicator values.

According to the numerical tests, the new multisection strategies resulted in substantial decrease both in the number of function evaluations and in the memory complexity.

II. The multisection strategy can also be applied for constrained global optimization problems [11]. The feasibility degree index for constraint $g_j(x) \leq 0$ can be formulated as

$$pu_{G_j}(X) = \min\left\{\frac{-\underline{G}_j(X)}{w(G_j(X))}, 1\right\}.$$

Notice that if $pu_{G_j}(X) < 0$ then the box is certainly infeasible, and if $pu_{G_j}(X) = 1$ then X certainly satisfies the constraint. Otherwise, the box is undetermined for that constraint. For boxes which are not certainly infeasible, i.e., for which $pu_{G_j}(X) \geq 0$ for all $j = 1, \ldots, r$ holds, the total infeasibility index is given by

$$pu(X) = \prod_{j=1}^{r} pu_{G_j}(X).$$

We must only define the index for such boxes since certainly infeasible boxes are immediately removed by the algorithm from further consideration. With this definition,

- $pu(X) = 1 \Leftrightarrow X$ is certainly feasible, and
- $pu(X) \in [0, 1) \Leftrightarrow X$ is undetermined.

Using the $pu(X)$ index, we now propose the following modification of the RejectIndex for constrained problems:

$$pup(\hat{f}, X) = pu(X) \cdot p(\hat{f}, X),$$

where $\hat{f}$ is a parameter of this indicator, which is usually an approximation of f^*. This new index works like $p(\hat{f}, X)$ if X is certainly feasible, but if the box is undetermined, then it takes the feasibility degree of the box into account: the less feasible the box is, the less the value of $pu(X)$ is.

A careful theoretical analysis proved that the new interval selection and multisection rules enable the branch-and-bound interval optimization algorithm to converge to a set of global optimizer points assumed we have a proper sequence of $\{f_k\}$ parameter values. The convergence properties obtained were very similar to those proven for the unconstrained case, and they give a firm basis for computational implementation.

A comprehensive numerical study on standard global optimization test problems and on facility location problems indicated [11] that the constrained version interval selection rules and to a less extent also the new adaptive multisection rules have several advantageous features that can contribute to the efficiency of the interval optimization techniques.

Heuristic rejection. RejectIndex can also be used to improve the efficiency of interval global optimization algorithms on very hard to solve problems by applying a rejection strategy to get rid of subintervals not containing global minimizer points. This *heuristic rejection* technique selects those subintervals on the basis of a typical pattern of changes in the pf^* values [3, 4].

The RejectIndex is not always reliable: assume that the inclusion function $F(X)$ of $f(x)$ is isotone, and α-convergent. Assume further that the RejectIndex parameter pf^* is less than 1 for all the subintervals of X. Then an arbitrary large number $N(> 0)$ of consecutive leading intervals may have the properties that:

i, neither of these processed intervals contains a stationary point, and

ii, during this phase of the search the pf^* values are maximal for these intervals as compared with the subintervals of the current working list.

Also, when a global optimization problem have a unique global minimizer point x^*, then there always exists an isotone and α-convergent

heuristic rejection

inclusion function $F(X)$ of $f(x)$ such that the new algorithm does not converge to x^*.

In spite of the possibility to lose the global minimum, obviously there exist such implementations that allow a safe way to use heuristic rejection. For example, the selected subintervals can be saved on a hard disk for further possible processing if necessary.

Although the above theoretical results were not encouraging, the computational tests on very hard global optimization problems were convincing: when the whole list of subintervals produced by the B&B algorithm is too large for the given computer memory, then the use of the suggested heuristic rejection technique decrease the number of working list elements without missing the global minimum. For hard to solve problems the new rejection test may make it possible to solve them, which are otherwise unsolvable with usual techniques.

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