Table of Contents

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Computer assisted proof of chaotic behaviour of the forced damped pendulum

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Abstract. J. Hubbard [4] has discovered that some motions of the damped forced pendulum

$$x'' + 10^{-1}x' + \sin x = \cos t$$

are chaotic in the sense that on the consecutive time intervals $(2k\pi, 2(k+1)\pi)$ $(k \in \mathbb{Z})$ they can freely "choose" between the following possibilities: the pendulum either crosses the bottom position clockwise exactly once or does not cross the bottom position at all or crosses the bottom position counterclockwise exactly once. (The pendulum does not point downwards at the time instants $t = 2k\pi,\ k \in \mathbb{Z}$.) We give a review of our work [1], in which we prove rigorously this assertion. The proof is based upon a theorem detecting chaos in general discrete dynamical systems. To check

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conditions of this theorem we need reliable computer simulations using the methods of interval arithmetic.

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1 Introduction

The mathematical pendulum is a material point of mass m hanging on a weightless rod of length ℓ in the gravitational field. The other end of the rod is fixed, and the rod can move in a plane (see Figure 1). Let x denote the angle measured

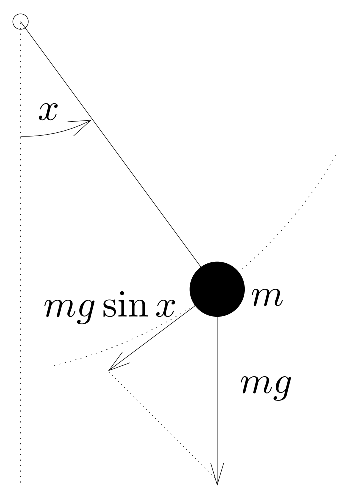


Fig. 1. Mathematical pendulum

counterclockwise from the direction of the gravity to the rod. If there acts also friction, i.e., the pendulum is damped, then, by Newton's Second Law, the motions of the pendulum are described by the second order differential equation

$$m\ell x'' = -mg\sin x - bx', \tag{1}$$

where g is the magnitude of the gravity, $b > 0$ is the damping coefficient; $x' = \mathrm{d}\,x/\mathrm{d}\,t$ denotes the angle velocity. The motions can be represented by trajectories on the phase plane (x, x') which are curves $t \mapsto (x(t), x'(t))$ belonging to solutions x of equation (1) (see Figure 2). There are asymptotically stable equilibria (sinks) $x = 2k\pi,\ x' = 0\ (k \in \mathbb{Z})$ at the downward position of the pendulum and unstable equilibria (saddles) $x = (2k+1)\pi,\ x' = 0$ at the upward position of the pendulum. The basins of the sinks are "vertical strips" separated by the ingoing curves of the

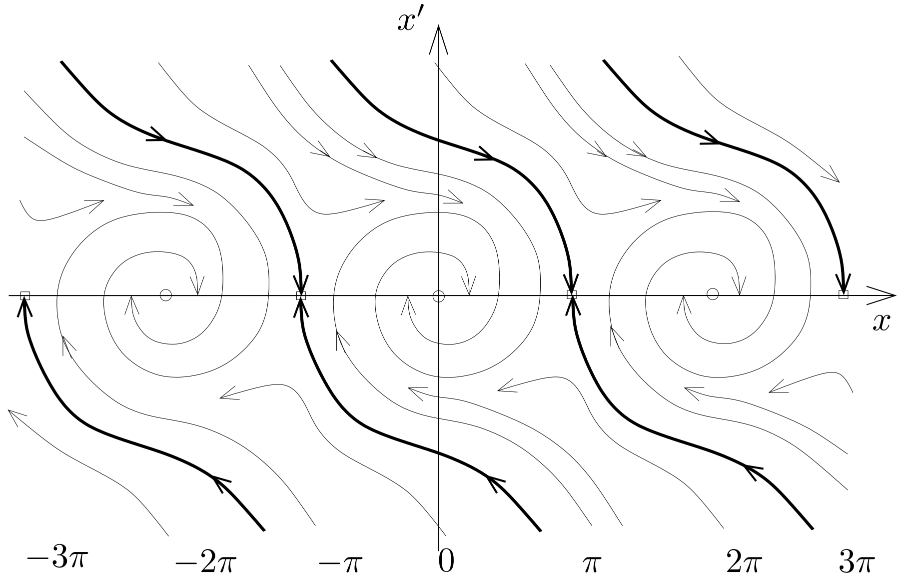


Fig. 2. Phase plane of the damped pendulum

saddles. We can say that almost all trajectories tend to sinks, and some exceptional trajectories tend to saddles as $t \to \infty$.

The situation becomes essentially more difficult when a periodic external force also acts on the point:

$$m\ell x'' = -mg \sin x - bx' + A \cos \omega t, \tag{2}$$

where A is the amplitude and ω is the frequency of the external force. R. Borelli and C. Coleman [2] observed that numerical solutions of equation (2) were very sensitive to the integration method, step-length, initial conditions near some points of the plane (x, x') at certain values of the parameters in the equation. We can illustrate this phenomenon integrating numerically the equation

$$x'' + 10^{-1}x' + \sin x = \cos t \tag{3}$$

starting from the three initial points

$$P_1(0,\ 1.98), \quad P_2(0,\ 2.00), \quad P_3(0,\ 2.01).$$

The $t - x$ graphs of the corresponding solutions can be seen on Figure 3. One has to observe that the solutions look asymptotically periodic with period 2π (period of the external force). This experiment suggests that there exists a stable periodic motion around the downward position, which ultimately attracts all the three solutions. This attracting periodic motion appears at different “levels” in our

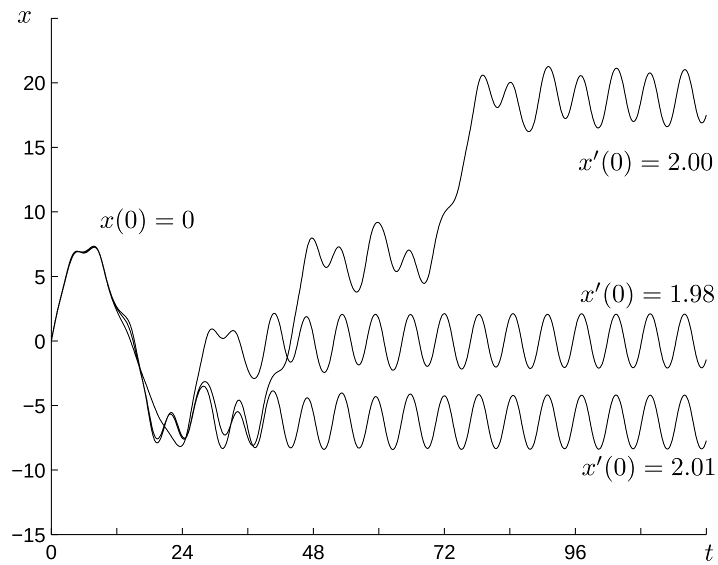


Fig. 3. Sensitivity to initial conditions

experiment; e.g., in the case of start point P_2 the motion goes "over the top" three times counterclockwise before settling down. After this experiment a superficial observer could think that a 2π- periodic solution attracts all solutions. But this is not true! J. Hubbard [4] has discovered uncountably many "strange" motions of the damped forced pendulum (3) whose asymptotic behaviour is unpredictable. He stated the following surprising result on the existence of chaos formulated by natural terms of the dynamics of such a natural mechanical system of one degree of freedom as the damped forced pendulum.

Theorem 1 (J.H. Hubbard [4]). *Suppose that there is a biinfinite sequence* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *given, where the numbers* $\varepsilon_k \in \{-1;0;1\}$ *are arbitrarily chosen. Then the forced damped pendulum described by equation (3) has at least one motion that corresponds to the biinfinite sequence* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *in the sense that during the time interval* $(2k\pi, 2(k+1)\pi)$ *the pendulum*

- *crosses the bottom position clockwise exactly once if and only if* $\varepsilon_k = -1$,
- *does not cross the bottom position at all if and only if* $\varepsilon_k = 0$,
- *crosses the bottom position counterclockwise exactly once if and only if* $\varepsilon_k = 1$,

and does not point downwards at the time instants $t = 2k\pi,\ k \in \mathbb{Z}$.

This theorem can be interpreted in the following way: ε_k is an event, so a biinfinite sequence $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ is an "itinerary" for the past and the future of the pendulum during a motion. The theorem says that for an arbitrary itinerary there exists a motion of the forced damped pendulum that during each time interval $(2k\pi, 2(k+1)\pi)$ will "do" ε_k. For example, during the motion corresponding to the itinerary $\{\dots, 0, 0, 0, \dots\}$ the pendulum *never* crosses the bottom position. In fact, we can prove that there exists an unstable 2π-periodic solution around the upper position not touching the bottom position.

Therefore, comparing the motions of the forced damped pendulum with those of the unforced damped pendulum we can say that, as an influence of the forcing, the stable and unstable *equilibria* of the unforced damped pendulum disappear, and a stable and an unstable *periodic solution* with the period of the forcing are born instead. The stable periodic solution attracts almost all motions, but there are exceptional motions, which are chaotic in some sense.

In [4] Hubbard did not prove Theorem 1. In a forthcoming paper we give a general theorem for detecting chaos in systems of differential equations, which can be applied to prove Theorem 1. The application of our theorem needs rigorous methods of computations, which will be done by interval arithmetic. By the same method we can prove the existence of a stable and an unstable periodic solution to equation (3). Here we give a review of these results.

2 The tools of the proof

Let $x(\cdot; t_0, x_0, x_0')$ denote the solution of (3) satisfying the initial condition $x(t_0; t_0, x_0, x_0') = x_0,\ x'(t_0; t_0, x_0, x_0') = x_0'$. The mapping

$$P : \mathbb{R}^2 \to \mathbb{R}^2,\ P : (x_0, x_0') \mapsto (x(2\pi; 0, x_0, x_0'),\ x'(2\pi; 0, x_0, x_0'))$$

is called the *period mapping* or *Poincaré mapping* to equation (3). If we are interested in stability properties of solutions of (3), then, instead of the *differential equation* (3), we can investigate the *discrete dynamical system*

$$P^k := \underbrace{P \circ P \circ \cdots \circ P}_{k-\text{times}} : \mathbb{R}^2 \to \mathbb{R}^2 \quad (k \in \mathbb{Z}). \tag{4}$$

An *orbit* of (4) is a biinfinite sequence

$$\{P^k(x_0, x_0')\}_{k\in\mathbb{Z}} \quad ((x_0, x_0') \in \mathbb{R}^2).$$

Solution $x(\cdot; 0, x_0, x_0')$ of (3) is 2π-periodic if and only if (x_0, x_0') is a fixed point of P. A 2π-periodic solution of (3) is stable if and only if the corresponding fixed point of P is stable in the discrete dynamical system (4).

J. Mawhin [5] gave sufficient conditions for the existence of periodic solutions to so-called "pendulum-like" second order differential equations. This theory guarantees at least one 2π-periodic solution to equation (3). We will prove that P has exactly two fixed points in the region $(0, 2\pi)\times(-\infty, \infty)$: a sink $s_0(4.236\dots, 0.392\dots)$ and a saddle $u_0(2.634\dots, 0.026\dots)$. The function $x \mapsto \sin x$ is 2π-periodic, so a horizontal 2π-shift of a fixed point of P is a fixed point, too. This means that we have infinitely many sinks and saddles:

$$s_k := s_0 + (2k\pi, 0), \quad u_k := u_0 + (2k\pi, 0) \quad (k \in \mathbb{Z}).$$

The basins of the sinks are of a very sophisticated structure. They are tangled; every basin meander around the plane. To be more precise: the basins seem to have the Wada property, i.e., every point of the boundary of any basin belongs to the boundaries of all others [4]. This is the root of the chaotic behaviour formulated in Theorem 1.

In the proof of Theorem 1 we will need certain quadrilaterals $\{Q_k\}_{k\in\mathbb{Z}}$ "long" in the unstable and "short" in the stable directions so that there are *"exceptional" orbits* of Poincaré mapping P with the following properties:

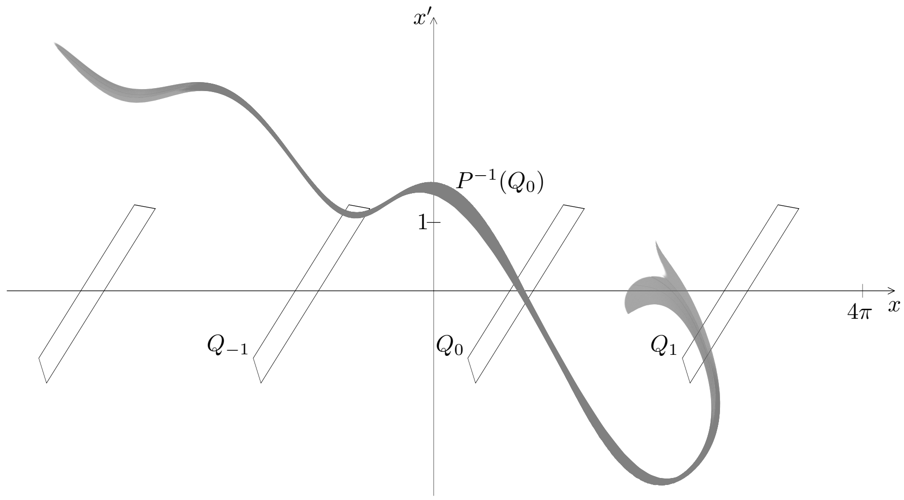
– an exceptional orbit is contained in $\cup_{k\in\mathbb{Z}} Q_k$;
– an exceptional orbit visits the quadrilaterals consecutively: if $P^n(x_0, x_0') \in Q_k$ for some $k, n \in \mathbb{Z}$, then either $P^{n+1}(x_0, x_0') \in Q_{k-1}$ or $P^{n+1}(x_0, x_0') \in Q_k$ or $P^{n+1}(x_0, x_0') \in Q_{k+1}$.

In the main step of the proof of Theorem 1 we will show that for an arbitrary consecutive order $\{Q_{i_k}\}_{k\in\mathbb{Z}}$ of quadrilaterals there is an exceptional orbit visiting the quadrilaterals in the prescribed order. To this end we have to know forward images $P(Q_k)$ and backward images $P^{-1}(Q_k)$. Thanks to the horizontal 2π-periodicity of the discrete dynamical system (4) it is enough to know the images $P(Q_0)$ and $P^{-1}(Q_0)$. For suitably chosen quadrilaterals the forward image $P(Q_0)$ crosses Q_{-1}, Q_0, Q_1 in long and thin "vertical strips", and the backward image $P^{-1}(Q_0)$ crosses Q_{-1}, Q_0, Q_1 in short and flat "horizontal strips" (see Figure 4). Let us denote these horizontal strips by R_{-1}, M_0, L_1, respectively. (P moves $R_{-1} \subset Q_{-1}$ to the *right*, it leaves the *middle* strip $M_0 \subset Q_0$ in Q_0, and it moves $L_1 \subset Q_1$ to the *left*.) The same connection is true for any triple Q_{k-1}, Q_k, Q_{k+1} ($k \in \mathbb{Z}$) with $R_{k-1} \subset Q_{k-1},\ M_k \subset Q_k,\ L_{k+1} \subset Q_{k+1}$. Using the method of interval arithmetic we can prove by reliable computer simulations that such $\{Q_k, R_k, M_k, L_k\}_{k\in\mathbb{Z}}$ exist.

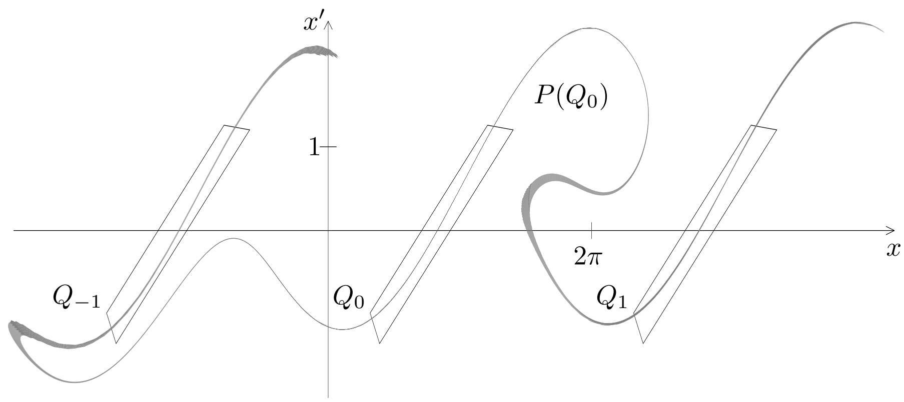
3 A topological theorem detecting chaos

Let us give rectangular sets $T_j = U_j \times S_j \subset \mathbb{R}^m \times \mathbb{R}^n$ (where $U_j \subset \mathbb{R}^m$ and $S_j \subset \mathbb{R}^n$ are compact topological balls, $j \in \mathbb{Z}$) and a continuous function $\phi : X := \cup_{j\in\mathbb{Z}} T_j \to \mathbb{R}^m \times \mathbb{R}^n$ whose coordinate functions are denoted by $\phi_u : X \to \mathbb{R}^m,\ \phi_s : X \to \mathbb{R}^n$. We define a so-called *transition graph* $\mathcal{G}(\phi)$ to these objects. The vertex set of $\mathcal{G}(\phi)$ is $\mathbb{Z}$, and $(j, j') \in \mathbb{Z}^2$ belongs to the edge set $\mathcal{E}(\phi)$ if and only if the following two conditions are satisfied:

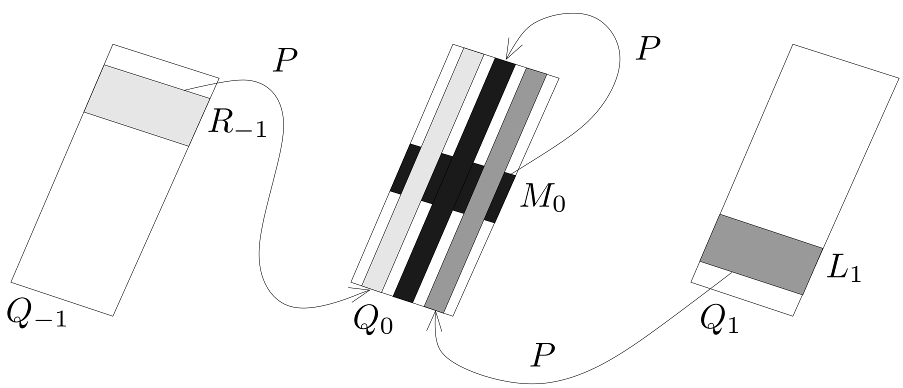
(i) $\phi(T_j) \subset (\mathbb{R}^m \times \mathbb{R}^n) \setminus (U_{j'} \times (\mathbb{R}^n \setminus S_{j'}))$ (ϕ contracts in the stable direction);



(a) Backward image of Q_0



(b) Forward image of Q_0



(c) Schematic $P(\cup_{k\in\mathbb{Z}}Q_k)\cap Q_0$

Fig. 4. Forward and backward images

(ii) set $\phi_u(\partial U_j \times S_j)$ "surrounds" set $U_{j'}$ (ϕ dilates in the unstable direction).

For example, on the plane ($m = n = 1$) $T_j, T_{j'}$ are real rectangles and the first condition says that the image of T_j has no point in the columns located "above" and "below" $T_{j'}$. The second condition means that the interval $U_{j'}$ is located between the projections onto the horizontal u-axis of the images of the two vertical sides of T_j. (The rather sophisticated precise analytical formulation of the geometrical condition (ii) can be found in [1].) $(j_0, \ldots, j_N) \in \mathbb{Z}^{N+1}$ ($N \geq 0$) is a *directed circle* in $\mathcal{G}(\phi)$ by definition if $(j_k, j_{k+1}) \in \mathcal{E}(\phi)$ ($k = 0, 1, \ldots, N; j_{N+1} := j_0$).

Theorem 2. *Suppose that $j_0, j_1, \ldots, j_N$ form a directed circle of $\mathcal{G}(\phi)$. Then there exist points $q_0, q_1, \ldots, q_N$ such that*

$$q_k \in T_{j_k}, \; \phi(q_k) = q_{k+1} \quad (k = 0, 1, \ldots, N; \; q_{N+1} := q_0).$$

In other words, for every directed circle $(j_0, j_1, \ldots, j_N)$ of the transition graph $\mathcal{G}(\phi)$ there exists an $(N+1)$-periodic orbit of ϕ visiting the rectangles of the circle in order $j_0, j_1, \ldots, j_N$.

The main tool of the proof (see [1]) is Brower's Fixed Point Theorem. Earlier proofs to similar results [8,7] were based on various topological index/degree arguments.

4 Sketch of the proof

We apply Theorem 2 to the rectangles

$$\{T_j\}_{j \in \mathbb{Z}} := \{\ldots, L_{-1}, M_{-1}, R_{-1}, L_0, M_0, R_0, L_1, M_1, L_1, \ldots\}$$

and to the Poincaré mapping P in (4). The edges of the corresponding transition graph $\mathcal{G}(P)$ can be deduced from Figure 4 (see Figure 5; since the graph is 3-

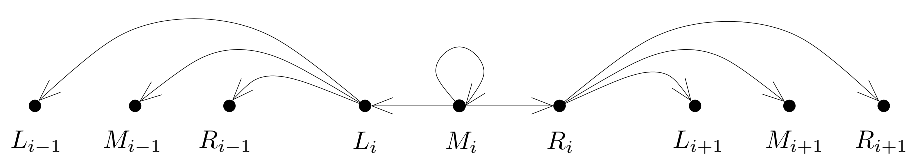


Fig. 5. The transition graph of the Poincaré mapping P

periodic, we drew in only the edges outgoing from L_i, M_i, and R_i.)

Let us given an itinerary

$$\{\ldots \varepsilon_{-3} \; \varepsilon_{-2} \; \varepsilon_{-1}. \; \varepsilon_0 \; \varepsilon_1 \; \varepsilon_2 \; \varepsilon_3 \ldots\}.$$

In the first step we realize a finite piece

$$\{\varepsilon_{-k}\ \varepsilon_{-k+1}\dots\varepsilon_{-1}.\ \varepsilon_0\ \varepsilon_1\dots\varepsilon_{k-1}\ \varepsilon_k\}\ (k\in\mathbb{Z}). \tag{5}$$

We illustrate the procedure by the example

$$\{-1\ \ -1\ 0\ .\ 1\ \ -1\ 0\ 0\}\quad (k=3). \tag{6}$$

We construct the chain of consecutive rectangles

$$\{T_{j_{-3}}\ T_{j_{-2}}\ T_{j_{-1}}.\ T_{j_0}\ T_{j_1}\ T_{j_2}\ T_{j_3}\}$$

to be visited by the desired orbit realizing (6). We start from Q_0. Since $\varepsilon_0 = 1$, we have to move to the right into Q_1; therefore, $T_{j_0} = R_0$. $\varepsilon_1 = -1$, hence we have to move from Q_1 to the left into Q_0; therefore, $T_{j_1} = L_1$. In the same way we get $T_{j_2} = M_0,\ T_{j_3} = M_0$. On the other hand, $\varepsilon_{-1} = 0$ means that during the time interval $[-2\pi, 0]$ the phase point has to move from Q_0 into R_0, so $T_{j_{-1}} = M_0$. Similarly, $T_{j_{-2}} = L_1,\ T_{j_{-3}} = L_2$, so the chain of rectangles sought for is

$$\{L_2\ L_1\ M_0\ .\ R_0\ L_1\ M_0\ M_0\}.$$

Since the transition graph $\mathcal{G}(P)$ is connected (see Figure 4), this chain can be completed by auxiliary edges into a circle:

$$\{L_2\ L_1\ M_0.\ R_0\ L_1\ M_0\ M_0\ \underbrace{R_0\ R_1}_{\text{auxiliary}}\ \},\quad (T_{j_4} := R_0,\ T_{j_5} := R_1)$$

Theorem 2 yields a periodic orbit $\{q_{j_i}\}_{i=-3}^{5}$ whose part $\{q_{j_i}\}_{i=-3}^{3}$ realizes (6). Similarly, we can construct a periodic orbit $\{q_{j_i}^k \in T_{j_i}\}_{i=-k}^{N_k}$ $(N_k \geq k)$ with a section realizing the finite itinerary (5). We can make the same procedure for all k's. But $q_{j_0}^k \in Q_0$ $(k \in \mathbb{Z})$ and Q_0 is compact; consequently, we may suppose that $\lim\limits_{k\to\infty} q_{i_0}^k =: q_0 \in Q_0$ exists. Then $\lim\limits_{k\to\infty} q_{j_i}^k =: q_i$ also exist for every $i \in \mathbb{Z}$, and $\{q_i\}_{i\in\mathbb{Z}}$ is the desired orbit realizing the itinerary $\{\varepsilon_i\}_{i\in\mathbb{Z}}$.

5 Reliable computer simulations

While applying Theorem 2 in the proof of Theorem 1 we essentially used the structure of transition graph $\mathcal{G}(P)$ (see Figure 5). To complete the proof we have to show that there are R_k, M_k, L_k $(k \in \mathbb{Z})$ such that $\mathcal{G}(P)$ has the edges drawn in Figure 4; e.g., $P(R_{-1}) \cap Q_0$ is a "long strip" in Q_0 crossing Q_0 in the way which can be seen in the figure. The problem is that the Poincaré mapping P is defined by solutions of differential equation (3). However, these solutions are not known, they have to be found numerically so that the assumed mutual positions of $P(R_{-1})$ and $R_0,\ M_0,\ L_0$ could be verify reliably. Using the method of interval arithmetic to find validated solutions of initial value problem for ODE's [6] we could prove

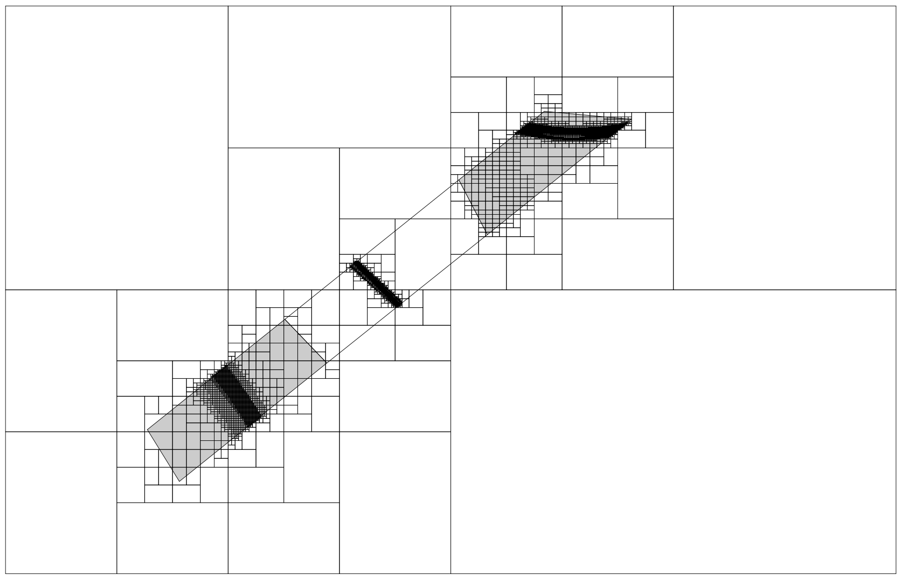


Fig. 6. Construction of R_0, M_0, and L_0

the existence of quadrilaterals Q_k ($k \in \mathbb{Z}$). The result is represented in Figure 6 (for details see [3,1]).

There is another step in the proof that needs validated computations. For example, to realize $\varepsilon_0 = 1$ in example (6) we choose $T_{j_0} = R_0,\ T_{j_1} = L_1$ guaranteeing that the solution curve $t \mapsto (x(t; 0, x_0, x_0'),\ x'(t; 0, x_0, x_0'))$ starts from Q_0 at $t = 0$ and meets Q_1 at $t = 2\pi$. Consequently, the curve crosses the vertical line $x = 2\pi$, i.e., the pendulum crosses the bottom position counterclockwise *at least once*. However, $\varepsilon_0 = 1$ means that this happens *exactly once*. To this end it is enough to show that if $(x_0, x_0') \in R_0$ and $x(2\pi; 0, x_0, x_0') \in Q_1$, then

$$\left.\begin{array}{l} x(t; 0, x_0, x_0') \neq 0,\ x(t; 0, x_0, x_0') \neq 4\pi \\ x(t; 0, x_0, x_0') = 2\pi \Rightarrow x'(t) > 0 \end{array}\right\} \quad (0 \leq t \leq 2\pi).$$

(Geometrically, the second condition excludes the behaviour shown by the broken curve in Figure 7). To show these properties we gave a reliable enclosure of the curves starting from R_0. Figure 8 shows one step of this procedure. While the small rectangles $a, b, \ldots$ contained in the large rectangles $A, B, C, \ldots$ give an enclosure for the points of the curve at certain finite values of $t \in [0, 2\pi]$, the large rectangles give an enclosure for the bundles of curves starting from the small rectangles. This enclosure guarantees the desired behaviour of solutions curves, and it completes the proof.

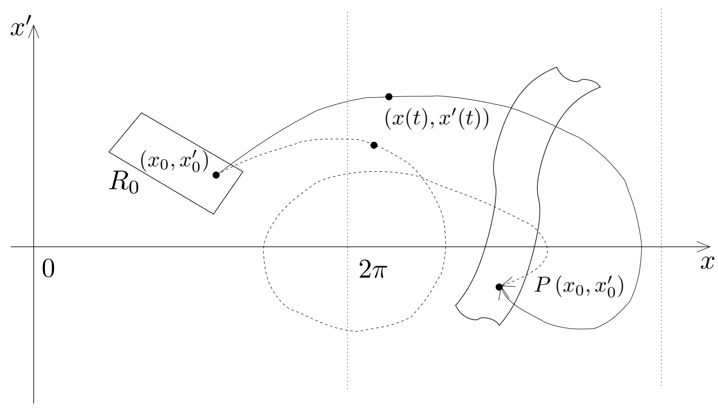


Fig. 7. The desired and excluded behaviour of solution curves

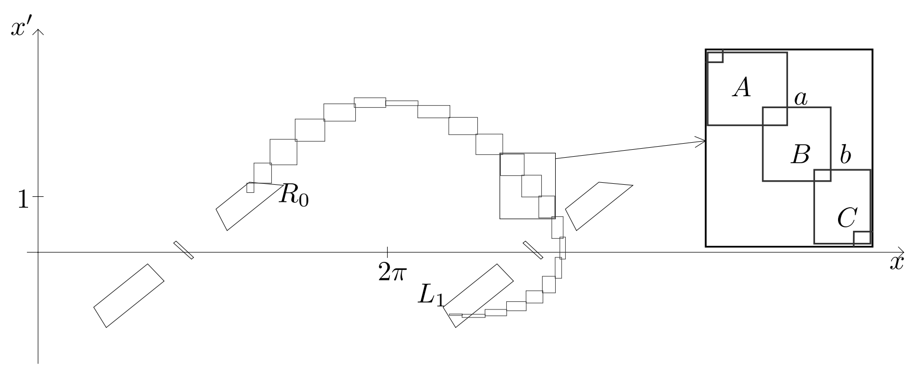


Fig. 8. Enclosure of solution curves

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